The linear affine functional equation and group actions.

Joint work with L. Reich and J. Schwaiger

Gronów, August 25, — September 1, 2002

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This equation and its solutions are known for instance

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– for r, x in the group $\mathbb{R}_{>0}$,
– for r, x in the semigroup $\mathbb{R}_{>1}$,
– on intervals,
– under certain regularity conditions (bounded on an interval).

First generalization

Let X be a set, $(R, \cdot)$ a group acting on X, and let V be a linear space over the field K.

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$$u(rx) = \alpha(r)u(x) + \beta(r) \qquad r \in R, x \in X \tag{1}$$

for the three unknown functions

$$u: X \to V \qquad \alpha: R \to K \qquad \beta: R \to V.$$

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A solution of (1) is indicated as a triple (u,α,β).

Later on we will also replace the vector space on the right hand side by the action of a (semi) group on a set Y.

Some special cases

Using methods both from the theory of functional equations and from group theory, we first deal with some special cases:

— $\alpha = 0$,

— u is constant,

— $\alpha = 1$,

— $\beta = 0$,

— $\alpha \neq 1$ a homomorphism, $\beta \neq 0$.

— — β satisfies the condition

$$\beta(rs) = \beta(sr) \qquad r, s \in R, \tag{2}$$

— — otherwise α is a homomorphism and (α, β) satisfies (4).

When is (2) satisfied?

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Lemma 6

Let α be a group homomorphism and assume that (α, β) is a solution of (4). There exists a solution (u, α, β) of (1) where u is constant on an orbit $\omega \in R\backslash\backslash X$, if and only if β satisfies (2).

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Corollary
If there exists an orbit $\omega \in R\backslash\backslash X$ of size 1, then for each solution (u, α, β) of (1), where α is a homomorphism, the function β satisfies (2).

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Example

If $X = R$ and R acts by conjugation on itself, then the conjugacy class of 1 consists of only one element, whence Lemma 5 can be applied.

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Let $K^* \times V$ be the affine group. That is, we consider $K^* \times V$ together with the multiplication $(\kappa_1, v_1)(\kappa_2, v_2) := (\kappa_1\kappa_2, v_1 + \kappa_1 v_2)$.

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We assume that these homomorphisms are known. Let $\varphi = (\alpha, \beta)$ be such a homomorphism. Introducing a function $\gamma\colon R \setminus \ker\alpha \to V$ given by $\gamma(r) := (1 - \alpha(r))^{-1}\beta(r)$, we can describe when there exists a function u such that (u, α, β) is a solution of (1). (I will not show the details here.)

Further generalizations

The vector space on the right hand side of (1) will be replaced by an arbitrary set Y on which a (semi)group S acts.

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The vector space on the right hand side of (1) will be replaced by an arbitrary set Y on which a (semi)group S acts. Then we want to solve the functional equation

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Clearly (3) is a generalization of (1).

Necessary conditions for solutions of (3)

If (u, φ) is a solution of (3), then

$$u(r_1 r_2 x) = \varphi(r_1 r_2) u(x) = \varphi(r_1)\varphi(r_2) u(x) \qquad r_1, r_2 \in R, x \in X.$$

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Lemma 9

For $x \in X$ we have $\psi(R_x) \subseteq S'_{u(x)}/\sim$.

The general solution of (3)

Theorem

Assume that $\mathcal{T}(R \backslash\backslash X)$ is given as $\{x_i \mid i \in I\}$.

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Choose a homomorphism $\psi\colon R \to S'/\sim$ with the properties

$$\psi(1) = \bar{e}, \qquad \psi(R_{x_i}) \subseteq S'_{y_i}/\sim.$$

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Choose a homomorphism $\psi\colon R \to S'/\sim$ with the properties

$$\psi(1) = \bar{e}, \qquad \psi(R_{x_i}) \subseteq S'_{y_i}/\sim.$$

If we put

$$\varphi\colon R \to S' : \varphi(r) \in \psi(r) \quad u\colon X \to Y : u(rx_i) = \psi(r)y_i,$$

then u is well defined and (u,φ) is a solution of (3).

Conclusion

The last Theorem describes the general solution of (3).

If we consider an action of a group (and not of a semigroup) on the right hand side of (1), then we can work with intersections of stabilizers, with normal subgroups and factor groups.

These general results can easily be rewritten for the particular action of the affine (semi)group on a vector space V since the stabilizer of a vector $v \in V$ has a very simple structure. The intersection of the stabilizers of two different vectors is just $\{1\}$.

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Lemma 1

Assume that $\alpha = 0$. Then we consider the equation

$$u(rx) = \beta(r) \qquad r \in R,\ x \in X.$$

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Lemma 1

Assume that $\alpha = 0$. Then we consider the equation

$$u(rx) = \beta(r) \qquad r \in R,\ x \in X.$$

The triple $(u, 0, \beta)$ is a solution of (1) if and only if there exists a vector $v \in V$ such that $u(x) = \beta(r) = v$ for all $x \in X$ and $r \in R$.

cont.

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Lemma 2

Assume that u is a constant function, say $u(x) = v$ for all $x \in X$.

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In order to construct all solutions (u, α, β) where u is constant, we can choose an arbitrary vector $v \in V$ and an arbitrary function α for the determination of β.

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In order to construct all solutions (u, α, β) where u is constant, we can choose an arbitrary vector $v \in V$ and an arbitrary function α for the determination of β.

If u is not constant, then for each solution (u, α, β) of (1) the function α is a group homomorphism from R to K^*, and β satisfies

$$\beta(rs) = \alpha(r)\beta(s) + \beta(r) \qquad r, s \in R. \tag{4}$$

cont.

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Lemma 3

Assume that $\alpha = 1$. Then we consider the equation

$$u(rx) = u(x) + \beta(r) \qquad r \in R,\ x \in X.$$

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Lemma 3

Assume that $\alpha = 1$. Then we consider the equation

$$u(rx) = u(x) + \beta(r) \qquad r \in R,\ x \in X.$$

Let $S := \langle r \in R \mid \exists x \in X : r \cdot x = x \rangle$. If $(u, 1, \beta)$ is a solution of (1), then β is a group homomorphism and $\ker \beta \geq S$.

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In order to construct all solutions $(u, 1, \beta)$ of (1) assume that β is a group homomorphism with $\ker \beta \geq S$. If u takes arbitrary values $u(x_0) \in V$ for x_0 belonging to a transversal $\mathcal{T}(R \backslash\backslash X)$ and $u(r \cdot x_0)$ is defined as $u(x_0) + \beta(r)$ for all $r \in R$, then $(u, 1, \beta)$ is a solution of (1).

cont.

Lemma 4

Assume that $\beta = 0$ and that u is not constant and consider the equation

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Let $S' := \langle r \in R \mid \exists x \in X : u(x) \neq 0 \text{ and } r \cdot x = x \rangle$. If $(u, \alpha, 0)$ is a solution of (1), then α is a homomorphism with $\ker \alpha \geq S'$. Moreover, for each orbit $\omega \in R \backslash\backslash X$, either $u(x) = 0$ for all $x \in \omega$ or $u(x) \neq 0$ for all $x \in \omega$.

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In order to construct all solutions $(u, \alpha, 0)$ of (1) where u is not constant, choose a subset X' of X different from X which is a union of R-orbits. Let $S'' := \langle r \in R \mid \exists x \in X \setminus X' : r \cdot x = x \rangle$, and let $\alpha \colon R \to K^*$ be a homomorphism with $\ker \alpha \geq S''$. If $u(x) = 0$ for all $x \in X'$ and if u takes arbitrary values $u(x_0) \in V \setminus \{0\}$ for all x_0 belonging to a transversal $\mathcal{T}(R \backslash\backslash (X \setminus X'))$ and $u(r \cdot x_0)$ is defined as $\alpha(r)u(x_0)$ for all $r \in R$, then $(u, \alpha, 0)$ is a solution of (1).

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If (u, α, β) is a solution of (1) where $\alpha \neq 1$ is a group homomorphism and β satisfies (2), then for any $s \notin \ker\alpha$ and for all $r \in R$

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holds, for some $v_0 \in V \setminus \{0\}$.

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The triple (u, α, β) is a solution of (1) with the given properties if and only if (U, α) is a solution of

$$U(rx) = \alpha(r)U(x) \tag{5}$$

where $U(x) := u(x) + v_0$ for all $x \in X$. (This equation was completely solved by Lemma 4.)

cont.

Group actions

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The orbit $R(x)$ is defined as $\{rx \mid r\in R\}$,
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the transversal $\mathcal{T}(R\backslash\backslash X)$ is a complete set of orbit representatives,
the stabilizer of x is $R_x=\{r\in R\mid rx=x\}$, a subgroup of R.

cont.

Semigroup actions

A multiplicative semigroup S with neutral element 1 acts on a set X if there exists a mapping

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Semigroup actions

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The stabilizer of x is $S_x=\{s\in S\mid sx=x\}$, a subsemigroup of S.

cont.

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