On formal cocycle equations

Harald Fripertinger
Karl-Franzens-Universität Graz
joint work with Ludwig Reich

ECIT 2010
September 12–17, 2010
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The formal translation equation for iteration groups of type II see H.F. and L. Reich: *The formal translation equation for iteration groups of type II*, Aequationes Math., 79: 111–156, 2010.

The translation equation

Translation equation

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We study solutions

$$F(s,x) = \sum_{n\geq 1} c_n(s)x^n \in \mathbb{C}[[x]]$$

of (T) in the ring of formal power series over $\mathbb{C}$ where $c_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq 1$, $c_1(s) \neq 0$, $s \in \mathbb{C}$.

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Solutions of (T) are called ***iteration groups***.

(T) implies $c_1(s+t) = c_1(s)c_1(t)$, $s,t \in \mathbb{C}$, whence c_1 is an exponential function.

Iteration groups of type I and II

Type I

If $c_1 \neq 1$, then $F(s,x)$ is of type I.

Then $c_n(s) = P_n(c_1(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$.

Iteration groups of type I and II

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If $c_1 \neq 1$, then $F(s,x)$ is of type I.
Then $c_n(s) = P_n(c_1(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$.

Type II

If $F(s,x) \neq x$ and if $c_1 = 1$, then $F(s,x)$ is of type II.
There exists an integer $k \geq 2$ so that $F(s,x) = x + c_k(s)x^k + \sum_{n>k} c_n(s)x^n$,
where $c_k(s+t) = c_k(s) + c_k(t)$, $s,t \in \mathbb{C}$, whence c_k is additive,
and $c_n(s) = P_n(c_k(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq k$.

The cocycle equations

In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(s,x))_{s\in\mathbb{C}}$ we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t,F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n.$$

Formal equations

If a is a nontrivial additive function, (or if e is a nontrivial exponential function) and if a polynomial relation $P\big(a(s),a(t)\big)=0$ (or $P\big(e(s),e(t)\big)=0$) holds true for all $s,t\in\mathbb{C}$, then $P=0$.

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This observations allows to study formal equations by replacing $a(s)$ and $a(t)$ (or $e(s)$ and $e(t)$) by indeterminates y and z.

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Moreover the mixed chain rule is valid for formal derivations.

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In $(\mathbb{C}[y])[[x]]$ we have the formal derivation with respect to x.

Moreover the mixed chain rule is valid for formal derivations.

Differentiation is now a purely algebraic process!

The formal translation equation

Formal translation equation in $(\mathbb{C}[y,z])[[x]]$ for iteration groups of type II

$$G(y+z,x) = G\big(y, G(z,x)\big) \qquad (\mathrm{T}_{\mathrm{formal}})$$

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k,$$

$$G(0,x) = x. \qquad (\mathrm{B})$$

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Theorem. Let $c_k \neq 0$ be an additive function. Then $F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of $(\mathrm{T}_{\mathrm{formal}})$ and (B).

The first cocycle equation

Let $F(s,x)$ be an iteration group. If $\alpha(s,x)=\sum_{n\geq 0}\alpha_n(s)x^n$ is a solution of $(\mathrm{Co1})$, then α_0 is an exponential function and

$$\hat{\alpha}(s,x):=\frac{\alpha(s,x)}{\alpha_0(s)}=1+\frac{\alpha_1(s)}{\alpha_0(s)}x+\cdots$$

is also a solution of $(\mathrm{Co1})$ and $(\mathrm{B1})$.

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The first cocycle equation

Let $F(s,x)$ be an iteration group. If $\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n$ is a solution of (Co1), then α_0 is an exponential function and

$$\hat{\alpha}(s,x) := \frac{\alpha(s,x)}{\alpha_0(s)} = 1 + \frac{\alpha_1(s)}{\alpha_0(s)}x + \cdots$$

is also a solution of (Co1) and (B1).

Substitution into the logarithmic series yields

$$\gamma(s,x) := \log\big(\hat{\alpha}(s,x)\big) = \sum_{n\geq 1} \gamma_n(s)x^n$$

satisfying

$$\gamma(s+t,x) = \gamma(s,x) + \gamma\big(t,F(s,x)\big) \qquad (\mathrm{Co1}_{\log})$$

together with $\gamma(0,x) = 0$.

If $F(s,x) = x + \sum_{n \geq k} P_n(c_k(s))x^n$ is an iteration group of type II, then each coefficient function $\gamma_n(s)$ is a polynomial $\tilde{P}_n(c_k(s))$ and for all $s,t \in \mathbb{C}$

$$\sum_{n \geq 1} \tilde{P}_n(c_k(s) + c_k(t))x^n = \sum_{n \geq 1} \tilde{P}_n(c_k(s))x^n + \sum_{n \geq 1} \tilde{P}_n(c_k(t)) \left[x + \sum_{r \geq k} P_r(c_k(s))x^r \right]^n .$$

If $F(s,x) = x + \sum_{n \geq k} P_n(c_k(s))x^n$ is an iteration group of type II, then each coefficient function $\gamma_n(s)$ is a polynomial $\tilde{P}_n(c_k(s))$ and for all $s,t \in \mathbb{C}$

$$\sum_{n\geq 1} \tilde{P}_n(c_k(s)+c_k(t))x^n = \sum_{n\geq 1} \tilde{P}_n(c_k(s))x^n + \sum_{n\geq 1} \tilde{P}_n(c_k(t)) \left[x + \sum_{r \geq k} P_r(c_k(s))x^r\right]^n .$$

This yields the formal first cocycle equation in $(\mathbb{C}[y,z])[[x]]$ for iteration groups of type II

$$\Gamma(y+z,x) = \Gamma(y,x) + \Gamma\big(z, G(y,x)\big) \qquad (\mathrm{Co1}_{\mathrm{formal}})$$

together with $\Gamma(0,x) = 0$ for $\Gamma(y,x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$, where $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a formal iteration group of type II.

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If $F(s,x) = x + \sum_{n\geq k} P_n(c_k(s))x^n$ is an iteration group of type II, then each coefficient function $\gamma_n(s)$ is a polynomial $\tilde{P}_n(c_k(s))$ and for all $s,t \in \mathbb{C}$

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This yields the formal first cocycle equation in $(\mathbb{C}[y,z])[[x]]$ for iteration groups of type II

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together with $\Gamma(0,x) = 0$ for $\Gamma(y,x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$, where $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a formal iteration group of type II.

Theorem. Let $c_k \neq 0$ be an additive function. Then $\gamma(s,x) = \sum_{n\geq 1} \tilde{P}_n(c_k(s))x^n$ is a solution of $(\mathrm{Co1}_{\log})$ satisfying $\gamma(0,x) = 0$ if and only if $\Gamma(y,x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$ is a solution of $(\mathrm{Co1}_{\mathrm{formal}})$ satisfying $\Gamma(0,x) = 0$.

Three equations derived from $(\mathrm{Co1}_{\mathrm{formal}})$

Differentiation of $(\mathrm{Co1}_{\mathrm{formal}})$ with respect to y and setting $y = 0$ yields

$$\frac{\partial}{\partial z}\Gamma(z,x) = K\big(\Gamma(z,x)\big), \qquad (\mathrm{Co1D}_{\mathrm{formal}})$$

where $K(x) := \frac{\partial}{\partial y}\Gamma(y,x)\big|_{y=0}$.

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Differentiation with respect to z together with the mixed chain rule yields

$$\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x), \qquad (\mathrm{Co1PD}_{\mathrm{formal}})$$

where $H(x) = \frac{\partial}{\partial y}G(y,x)|_{y=0}$ is the formal generator of the formal iteration group $G(y,x)$.

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Three equations derived from $(\mathrm{Co1}_{\mathrm{formal}})$

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$$\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x), \qquad (\mathrm{Co1PD}_{\mathrm{formal}})$$

where $H(x) = \frac{\partial}{\partial y}G(y,x)|_{y=0}$ is the formal generator of the formal iteration group $G(y,x)$. There is also an Aczél–Jabotinsky differential equation

$$K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x) = K\big(\Gamma(y,x)\big). \qquad (\mathrm{Co1AJ}_{\mathrm{formal}})$$

Theorem. Each solution of $(\mathrm{Co1PD_{formal}})$ or $(\mathrm{Co1D_{formal}})$ together with $\Gamma(0,x)=0$ is a solution of $(\mathrm{Co1_{formal}})$.

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Theorem. Let $K(x) = \sum_{n\geq 1} \kappa_n x^n$ be a formal series of order at least 1. Then the solution of (Co1D$_{\text{formal}}$) with $\Gamma(0,x) = 0$ is

$$\Gamma(y,x) = \sum_{j=1}^{k-1} \int \kappa_j \left[G(\sigma,x)\right]^j d\sigma + E\big(G(y,x)\big) - E(x),$$

where $E(x)$ is given by $\dfrac{\partial}{\partial x} E(x) = \dfrac{\sum_{n\geq k} \kappa_n x^n}{H(x)}$.

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where $E(x)$ is given by $\frac{\partial}{\partial x}E(x)=\frac{\sum_{n\geq k}\kappa_n x^n}{H(x)}$.

By applying the exponential series we obtain the solutions of (Co1) as

$$\alpha_1(s)\frac{\tilde{E}\big(G(c_k(s),x)\big)}{\tilde{E}(x)}\prod_{j=1}^{k-1}\exp\Big(\int \kappa_j[G(\sigma,x)]^j d\sigma|_{\sigma=c_k(s)}\Big),$$

where $\tilde{E}(x)=\exp\big(E(x)\big)=1+\ldots$.

Theorem. Let $K(x) = \sum_{n\geq 1} \kappa_n x^n$ be a formal series of order at least 1. The polynomials $\tilde{P}_n(y)$ describing the coefficients of the solution of $(\mathrm{Co1D}_{\mathrm{formal}})$ and $\Gamma(0,x) = 0$ are universal polynomials of the form

$$\tilde{P}_n(y) = \begin{cases} \kappa_n y & n < k \\ \kappa_k y + \frac{\kappa_1}{2} y^2 & n = k \\ \kappa_n y + \frac{(n-k+1)\kappa_{n-k+1}}{2} y^2 + \tilde{Q}_n(y, \kappa_1, \ldots, \kappa_{n-k}) & n > k, \end{cases}$$

and of a formal degree $1 + \lfloor \frac{n-1}{k-1} \rfloor$.

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and of a formal degree $1 + \lfloor \frac{n-1}{k-1} \rfloor$.

Similarities to the solutions $G(y,x) = x + yx^k + \sum_{n>k} P_n(y) x^n$ of $(\mathrm{T}_{\mathrm{formal}})$ of type II depending on the formal generator $H(x) = \sum_{n\geq k} h_n x^n$:
The polynomials $P_n(y)$ are universal polynomials of the form

$$P_n(y) = \begin{cases} h_n y & \text{if } k \leq n < 2k-1 \\ h_{2k-1} y + \frac{k}{2} y^2 & \text{if } n = 2k-1 \\ h_n y + \frac{n+1}{2} h_{n-k+1} y^2 + \Phi_n(y, h_{k+1}, \ldots, h_{n-k}) & \text{if } n \geq 2k, \end{cases}$$

and of a formal degree $\lfloor \frac{n-1}{k-1} \rfloor$.

Reordering the summands

Solution of $(\text{Co1}_{\text{formal}})$: $\Gamma(y,x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n \in (\mathbb{C}[y])[[x]]$

$$\tilde{P}_n(y) = \sum_{j=1}^{d_n} \tilde{P}_{n,j} y^j \in \mathbb{C}[y], \quad d_n = 1 + \left\lfloor \frac{n-1}{k-1} \right\rfloor, \qquad n \geq 1,$$

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$$\Gamma(y,x) = \sum_{n\geq 1} \psi_n(x) y^n \in (\mathbb{C}[[x]])[[y]]$$

$$\psi_n(x) = \sum_{r\geq 1} \tilde{P}_{r,n} x^r, \qquad n \geq 1.$$

$\left(\psi_n(x)\right)_{n\geq 1}$ and $\left(\psi_n(x)y^n\right)_{n\geq 1}$ are summable families.

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$\left(\psi_n(x)\right)_{n\geq 1}$ and $\left(\psi_n(x) y^n\right)_{n\geq 1}$ are summable families.

This allows us to rewrite $(\mathrm{Co1PD}_{\mathrm{formal}})$ as

$$\sum_{n\geq 1} n \psi_n(x) y^{n-1} = K(x) + H(x) \sum_{n\geq 1} \psi_n'(x) y^n. \tag{1}$$

(1) is satisfied if and only if

$$\psi_1(x) = K(x)$$

$$\psi_{n+1}(x) = \frac{1}{n+1} H(x)\psi_n'(x), \quad n \geq 1,$$

holds true.

$$\psi_1(x) = K(x),$$

$$\psi_2(x) = H(x)K'(x)/2,$$

$$\psi_3(x) = \big(H(x)H'(x)K'(x) + H(x)^2 K''(x)\big)/6.$$

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Similarities to formal iteration groups $G(y,x) = \sum_{n\geq 0} \phi_n(x) y^n$ of type II:

$$\phi_{n+1}(x) = \frac{1}{n+1} H(x)\phi_n'(x), \quad n \geq 0.$$

$$\phi_1(x) = H(x),$$

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Some results

Generalizing the representations given on the previous slide:

$$\psi_n(x) = \frac{1}{n!} \sum_{i \in J_n} L(i) K^{(i_{-1})}(x) \prod_{j=0}^{n-2} \left[H^{(j)}(x) \right]^{i_j}, \qquad n \geq 2,$$

$$J_n = \left\{ (i_j)_{j \geq -1} \,\middle|\, i_{-1} \geq 1,\ i_j \geq 0,\ \sum_{j \geq 0} i_j = n-1,\ \sum_{j \geq 1} j i_j = n-1-i_{-1} \right\},$$

where $L\colon \bigcup_{n \geq 2} J_n \to \mathbb{N}$ is recursively determined.

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where $L\colon \bigcup_{n \geq 2} J_n \to \mathbb{N}$ is recursively determined.

Similarities to iteration groups of type II:

$$\phi_n(x) = \frac{1}{n!} \sum_{i \in I_n} L(i) \prod_{j=0}^{n-1} \left[H^{(j)}(x) \right]^{i_j}, \qquad n \geq 1,$$

$$I_n = \left\{ (i_j)_{j \geq 0} \,\middle|\, i_j \geq 0,\ i_0 \geq 1,\ \sum_{j \geq 0} i_j = n,\ \sum_{j \geq 1} j i_j = n-1 \right\},$$

where $L\colon \bigcup_{n \geq 1} I_n \to \mathbb{N}$ is recursively determined.

Solution as a Lie–Gröbner-series

$$\Gamma(y,x) := \sum_{n\geq 1} \frac{1}{n!} D^{n-1}\big(K(x)\big) y^n,$$

where

$$D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D\big(f(x)\big) := H(x) f'(x).$$

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Similarities to iteration groups of type II:

$$G(y,x) := \sum_{n\geq 0} \frac{1}{n!} D^n(x)y^n,$$

where

$$D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D\big(f(x)\big) := H(x)f'(x).$$

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