

## On a functional equation involving group actions

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**Summary.** During the forty-first ISFE in Noszvaj, Hungary, G. Guzik posed a problem on a functional equation involving group actions which arose in a generalization of Bargman theory occurring in Quantum Mechanics. (Cf. 18. Problem and Remark in “Report of Meeting”, Aequationes Math., 67 (2004), 312–313.)

Let  $(G, \cdot)$  be a group which is acting on a set  $X$  and let  $(K, +)$  be an abelian group. Describe all functions  $f: G \times G \times X \rightarrow K$  satisfying

$$f(g_1, g_2, x) + f(g_1g_2, g_3, x) = f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x)$$

for all  $g_1, g_2, g_3 \in G$  and  $x \in X$ .

This problem was solved in a particular case by B. Ebanks. (Cf. 19. Remark in “Report of Meeting”, Aequationes Math. 67 (2004), p. 313.) We present the general solution of this problem.

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### 1. Introduction

During the forty-first ISFE in Noszvaj, Hungary, G. Guzik posed a problem on a functional equation involving group actions which arose in a generalization of Bargman theory occurring in Quantum Mechanics (cf. [4]). This problem was solved in a particular case by B. Ebanks [2] during the conference. We present the general solution of this problem.

**Problem.** Let  $(G, \cdot)$  be a group which is acting on a set  $X$  and let  $(K, +)$  be an abelian group. Describe all functions  $f: G \times G \times X \rightarrow K$  satisfying

$$f(g_1, g_2, x) + f(g_1g_2, g_3, x) = f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x) \quad (1)$$

for all  $g_1, g_2, g_3 \in G$  and  $x \in X$ .

In the functional equation (1) all arguments in the third place of  $f$  belong to one orbit only. If  $f$  is a solution of (1), then the restriction of  $f$  to  $G \times G \times G(x_0)$ ,

where  $G(x_0)$  is the orbit of an element  $x_0 \in X$ , is a solution of (1) on  $G \times G \times G(x_0)$ . If, conversely, we know solutions  $f_\omega$  of (1) on  $G \times G \times \omega$  for all orbits  $\omega$  of  $G$ , then it is obvious how to obtain a solution of (1) on  $G \times G \times X$ . Hence, without loss of generality, we assume that  $G$  acts transitively on  $X$ , i.e.  $X = G(x_0)$  consists of one orbit only.

Let  $H$  be the stabilizer of  $x_0$ , i.e.

$$H = \{g \in G \mid gx_0 = x_0\},$$

which is a subgroup of  $G$ .

## 2. Two particular cases

First we consider the two situations  $H = \{1\}$  and  $H = G$ .

If  $H = \{1\}$ , then the sets  $G(x_0)$  and  $G$  are one-to-one. A bijection is given by

$$G(x_0) \ni gx_0 \mapsto g \in G.$$

Therefore we rewrite (1) as

$$f(g_1, g_2, g) + f(g_1g_2, g_3, g) = f(g_2, g_3, g_1^{-1}g) + f(g_1, g_2g_3, g) \quad (2)$$

for all  $g_1, g_2, g_3, g \in G$ . During the forty-first ISFE in Noszvaj, Hungary, B. Ebanks proved the following theorem:

**Theorem 1.** *The general solution of (2) is given by*

$$f(g_1, g_2, g_3) = \varphi(g_3^{-1}, g_1) + \varphi(g_3^{-1}g_1, g_2) - \varphi(g_3^{-1}, g_1g_2), \quad g_1, g_2, g_3 \in G, \quad (3)$$

where  $\varphi$  is an arbitrary mapping from  $G \times G$  to  $K$ .

*Proof.* If  $f$  is a solution of (2) then by putting  $g = 1$  and  $\varphi(g_1, g_2) := f(g_1, g_2, 1)$ ,  $g_1, g_2 \in G$ , we obtain

$$f(g_2, g_3, g_1^{-1}) = \varphi(g_1, g_2) + \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3), \quad g_1, g_2, g_3 \in G.$$

Conversely, direct computations show that each  $f$  given by (3) is a solution of (2).  $\square$

If  $H = G$ , then  $G(x_0) = \{x_0\}$ , whence the third component of  $f$  can be omitted. We therefore derive from (1) the *cocycle equation*

$$f(g_1, g_2) + f(g_1g_2, g_3) = f(g_2, g_3) + f(g_1, g_2g_3) \quad (4)$$

for all  $g_1, g_2, g_3 \in G$ .

If  $G$  is abelian and  $K$  is a divisible abelian group, which is uniquely divisible by 2, then the general solution of (4) is given by

$$f(g_1, g_2) = \psi(g_1) + \psi(g_2) - \psi(g_1g_2) + \Psi(g_1, g_2), \quad g_1, g_2 \in G, \quad (5)$$

where  $\psi$  is an arbitrary mapping from  $G$  to  $K$  and  $\Psi$  is a mapping from  $G \times G \rightarrow K$  so that  $\Psi(g_1, g_2) = -\Psi(g_2, g_1)$  and  $\Psi(g_1g'_1, g_2) = \Psi(g_1, g_2) + \Psi(g'_1, g_2)$  for all  $g_1, g'_1, g_2 \in G$ . (See e.g. [7], [3], [5], [1], [6].)

If  $G$  is an arbitrary group, then each  $f$  of the form (5) is a solution of the cocycle equation.

### 3. The general case

Assume that the stabilizer  $H$  of  $x_0$  is an arbitrary subgroup of  $G$ . Let  $f$  be a solution of (1), then for all  $g_1, g_2, g_3 \in G$  we have

$$f(g_2, g_3, g_1^{-1}x_0) = f(g_1, g_2, x_0) + f(g_1g_2, g_3, x_0) - f(g_1, g_2g_3, x_0).$$

Define the function  $\varphi: G \times G \rightarrow K$  by  $\varphi(g_1, g_2) := f(g_1, g_2, x_0)$ . Thus we obtain

$$f(g_2, g_3, g_1^{-1}x_0) = \varphi(g_1, g_2) + \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3), \quad g_1, g_2, g_3 \in G. \quad (6)$$

If  $g_1 \in H$ , then  $g_1^{-1}x_0 = x_0$  and consequently

$$\varphi(g_2, g_3) = \varphi(h_1, g_2) + \varphi(h_1g_2, g_3) - \varphi(h_1, g_2g_3), \quad h_1 \in H, g_2, g_3 \in G. \quad (7)$$

In particular

$$\varphi(h_2, h_3) = \varphi(h_1, h_2) + \varphi(h_1h_2, h_3) - \varphi(h_1, h_2h_3), \quad h_1, h_2, h_3 \in H,$$

which is the cocycle equation on  $H$ . Therefore the restriction of  $\varphi$  to  $H \times H$  is a solution of the cocycle equation on  $H$ . Putting  $h_1 := h$  and  $h_2 := h_3 := 1$  we obtain that  $\varphi(h, 1) = \varphi(1, 1)$  for all  $h \in H$  and all solutions  $\varphi$  of the cocycle equation.

Let  $(\gamma_i)_{i \in I}$  with  $\gamma_0 = 1$  be a complete system of representatives of the right cosets of  $H$  in  $G$ . Consequently

$$G = \dot{\bigcup}_{i \in I} H\gamma_i.$$

In (7) we set  $g_2 := h_2 \in H$  and  $g_3 := \gamma_i$  for some  $i \in I$ . Then we get

$$\varphi(h_1, h_2\gamma_i) = \varphi(h_1, h_2) + \varphi(h_1h_2, \gamma_i) - \varphi(h_2, \gamma_i).$$

For  $i \in I$  define  $\varphi_i: H \rightarrow K$  by  $\varphi_i(h) := \varphi(h, \gamma_i)$ . Then we derive

$$\varphi(h_1, h_2\gamma_i) = \varphi(h_1, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2), \quad h_1, h_2 \in H, i \in I.$$

Especially  $\varphi_0(h) = \varphi(h, 1) = \varphi(1, 1)$  for  $h \in H$ .

**Lemma 2.** *The function  $\varphi: H \times G \rightarrow K$  is a solution of*

$$\varphi(h_2, g_3) = \varphi(h_1, h_2) + \varphi(h_1h_2, g_3) - \varphi(h_1, h_2g_3), \quad h_1, h_2 \in H, g_3 \in G, \quad (7')$$

*if and only if there exists a solution  $\phi: H \times H \rightarrow K$  of the cocycle equation on  $H$  and functions  $\varphi_i: H \rightarrow K$  which are arbitrary for  $i \in I \setminus \{0\}$  and  $\varphi_0(h) = \phi(1, 1)$ ,  $h \in H$ , so that*

$$\varphi(h_1, h_2\gamma_i) = \phi(h_1, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2), \quad h_1, h_2 \in H, i \in I. \quad (8)$$

*Therefore,  $\varphi$  is an extension of  $\phi$ .*

*Proof.* If  $\varphi$  satisfies (7'), then we have just shown that  $\varphi$  has the desired representation.

If  $\varphi$  is given by (8), then  $\varphi$  is an extension of  $\phi$ ,

$$\begin{aligned}\varphi(h_1, h_2) &= \varphi(h_1, h_2\gamma_0) \\ &= \phi(h_1, h_2) + \varphi_0(h_1h_2) - \varphi_0(h_2) \\ &= \phi(h_1, h_2) + \phi(1, 1) - \phi(1, 1) \\ &= \phi(h_1, h_2), \quad h_1, h_2 \in H.\end{aligned}$$

Now we prove that  $\varphi$  satisfies (7'). Consider  $h_1, h_2 \in H$  and  $g_3 \in G$ . Then there exists some  $i \in I$  and  $h_3 \in H$  so that  $g_3 = h_3\gamma_i$  and

$$\begin{aligned}&\varphi(h_1, h_2) + \varphi(h_1h_2, h_3\gamma_i) - \varphi(h_1, h_2h_3\gamma_i) \\ &\stackrel{(8)}{=} \phi(h_1, h_2) + \phi(h_1h_2, h_3) + \varphi_i(h_1h_2h_3) - \varphi_i(h_3) - \phi(h_1, h_2h_3) - \varphi_i(h_1h_2h_3) \\ &\quad + \varphi_i(h_2h_3) \\ &= \phi(h_1, h_2) + \phi(h_1h_2, h_3) - \phi(h_1, h_2h_3) + \varphi_i(h_2h_3) - \varphi_i(h_3) \\ &\stackrel{(4)}{=} \phi(h_2, h_3) + \varphi_i(h_2h_3) - \varphi_i(h_3) \\ &\stackrel{(8)}{=} \varphi(h_2, h_3\gamma_i)\end{aligned}$$

which finishes the proof.  $\square$

Again we assume that  $\varphi$  satisfies (7). In (7) we set  $g_2 := \gamma_i$  for some  $i \in I$ . Then we get

$$\varphi(h_1\gamma_i, g_3) = \varphi(\gamma_i, g_3) + \varphi(h_1, \gamma_i g_3) - \varphi(h_1, \gamma_i).$$

For  $i \in I$  define  $\theta_i: G \rightarrow K$  by  $\theta_i(g) := \varphi(\gamma_i, g)$ . Then we derive

$$\varphi(h_1\gamma_i, g_3) = \theta_i(g_3) + \varphi(h_1, \gamma_i g_3) - \varphi(h_1, \gamma_i), \quad h_1 \in H, i \in I, g_3 \in G.$$

Especially  $\theta_0(g) = \varphi(1, g) = \varphi(1, 1)$  for  $g \in G$  (which follows from (7) by setting  $h_1 := g_2 := 1$  and  $g_3 := g$ ).

**Lemma 3.** *The function  $\varphi: G \times G \rightarrow K$  is a solution of (7) if and only if there exists a solution  $\phi: H \times G \rightarrow K$  of (7') and functions  $\theta_i: G \rightarrow K$  which are arbitrary for  $i \in I \setminus \{0\}$  and  $\theta_0(g) = \phi(1, 1)$ ,  $g \in G$ , so that*

$$\varphi(h_1\gamma_i, g_3) = \theta_i(g_3) + \phi(h_1, \gamma_i g_3) - \phi(h_1, \gamma_i), \quad h_1 \in H, i \in I, g_3 \in G. \quad (9)$$

Therefore,  $\varphi$  is an extension of  $\phi$ .

*Proof.* If  $\varphi$  satisfies (7), then we have just shown that  $\varphi$  has the desired representation.

If  $\varphi$  is given by (9), then  $\varphi$  is an extension of  $\phi$ , since

$$\begin{aligned}
\varphi(h, g) &= \varphi(h\gamma_0, g) \\
&= \theta_0(g) + \phi(h, g) - \phi(h, \gamma_0) \\
&= \phi(1, 1) + \phi(h, g) - \phi(h, 1) \\
&= \phi(h, g), \quad h \in H, g \in G.
\end{aligned}$$

Now we prove that  $\varphi$  satisfies (7). Consider  $h_1 \in H$  and  $g_2, g_3 \in G$ . Then there exist  $i, j \in I$  and  $h_2, h_3 \in H$  so that  $g_2 = h_2\gamma_i$  and  $g_3 = h_3\gamma_j$ . Moreover  $\gamma_i h_3\gamma_j = h_4\gamma_k$  for some  $h_4 \in H$  and  $k \in I$ . We obtain

$$\begin{aligned}
&\varphi(h_1, h_2\gamma_i) + \varphi(h_1h_2\gamma_i, h_3\gamma_j) - \varphi(h_1, h_2\gamma_i h_3\gamma_j) \\
&\stackrel{(9)}{=} \phi(h_1, h_2\gamma_i) + \theta_i(h_3\gamma_j) + \phi(h_1h_2, \gamma_i h_3\gamma_j) - \phi(h_1h_2, \gamma_i) - \phi(h_1, h_2h_4\gamma_k) \\
&\stackrel{(8)}{=} \phi(h_1, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2) + \theta_i(h_3\gamma_j) + \phi(h_1h_2, h_4) + \varphi_k(h_1h_2h_4) \\
&\quad - \varphi_k(h_4) - \phi(h_1h_2, 1) - \varphi_i(h_1h_2) + \varphi_i(1) - \phi(h_1, h_2h_4) - \varphi_k(h_1h_2h_4) \\
&\quad + \varphi_k(h_2h_4) \\
&= \phi(h_1, h_2) + \phi(h_1h_2, h_4) - \phi(h_1, h_2h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4) \\
&\quad - \varphi_i(h_2) - \phi(1, 1) + \varphi_i(1) \\
&\stackrel{(4)}{=} \phi(h_2, h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \varphi_i(h_2) - \phi(1, 1) + \varphi_i(1).
\end{aligned}$$

And

$$\begin{aligned}
&\varphi(g_2, g_3) \\
&= \varphi(h_2\gamma_i, h_3\gamma_j) \\
&\stackrel{(9)}{=} \theta_i(h_3\gamma_j) + \phi(h_2, \gamma_i h_3\gamma_j) - \phi(h_2, \gamma_i) \\
&\stackrel{(8)}{=} \theta_i(h_3\gamma_j) + \phi(h_2, h_4) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \phi(h_2, 1) - \varphi_i(h_2) + \varphi_i(1) \\
&= \phi(h_2, h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \varphi_i(h_2) - \phi(1, 1) + \varphi_i(1)
\end{aligned}$$

which finishes the proof.  $\square$

**Corollary 4.** Let  $(G, \cdot)$  be a group,  $H$  a subgroup of  $G$ ,  $(\gamma_i)_{i \in I}$  with  $\gamma_0 = 1$  a complete system of representatives of the right cosets of  $H$  in  $G$ , and  $(K, +)$  an abelian group. The function  $\varphi: G \times G \rightarrow K$  is a solution of (7) if and only if  $\varphi$  restricted to  $H \times H$  satisfies the cocycle equation and

$$\begin{aligned}
\varphi(h_1, h_2\gamma_i) &= \varphi(h_1, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2) \quad h_1, h_2 \in H, i \in I, \\
\varphi(h_1\gamma_i, g_3) &= \theta_i(g_3) + \varphi(h_1, \gamma_i g_3) - \varphi(h_1, \gamma_i) \quad h_1 \in H, i \in I, g_3 \in G.
\end{aligned}$$

where  $\varphi_i: H \rightarrow K$  and  $\theta_i: G \rightarrow K$ ,  $i \in I$ , are arbitrary functions with exception of  $\varphi_0 = \theta_0 = \varphi(1, 1)$ .

**Lemma 5.** *Let  $G$  act transitively on  $X$ , take  $x_0 \in X$  fixed, let  $H$  be the stabilizer of  $x_0$  and assume that for  $x \in X$*

$$x = g_1^{-1}x_0 = \tilde{g}_1^{-1}x_0$$

hence  $\tilde{g}_1 = hg_1$  for some  $h \in H$ . Let  $\varphi$  be a solution of (7), then

$$\varphi(g_1, g_2) + \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3) = \varphi(\tilde{g}_1, g_2) + \varphi(\tilde{g}_1g_2, g_3) - \varphi(\tilde{g}_1, g_2g_3).$$

*Proof.* We prove that

$$\varphi(g_1, g_2) + \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3) = \varphi(hg_1, g_2) + \varphi(hg_1g_2, g_3) - \varphi(hg_1, g_2g_3)$$

for all  $g_1, g_2, g_3 \in G$  and  $h \in H$ . This will be done by showing that the right hand side does not depend on  $h$ , so we can take  $h = 1$  and obtain the left hand side.

Let  $(\gamma_i)_{i \in I}$  be a complete system of representatives of the right cosets of  $H$  in  $G$  with  $\gamma_0 = 1$ . Then there exist  $h_1, \dots, h_6 \in H$  and  $\gamma_{(1)}, \gamma_{(2)}, \gamma_{(3)}, \gamma_{(12)}, \gamma_{(23)}, \gamma_{(123)}$  elements of  $(\gamma_i)_{i \in I}$  so that  $g_1 = h_1\gamma_{(1)}, g_2 = h_2\gamma_{(2)}, g_3 = h_3\gamma_{(3)}$  and

$$\begin{aligned} \gamma_{(1)}h_2\gamma_{(2)} &= h_4\gamma_{(12)} \\ \gamma_{(2)}h_3\gamma_{(3)} &= h_5\gamma_{(23)} \\ \gamma_{(1)}h_2\gamma_{(2)}h_3\gamma_{(3)} &= h_6\gamma_{(123)} \\ \gamma_{(12)}h_3\gamma_{(3)} &= h_4^{-1}\gamma_{(1)}h_2\gamma_{(2)}h_3\gamma_{(3)} \\ &= h_4^{-1}h_6\gamma_{(123)} \\ \gamma_{(1)}h_2h_5\gamma_{(23)} &= \gamma_{(1)}h_2\gamma_{(2)}h_3\gamma_{(3)} \\ &= h_6\gamma_{(123)}. \end{aligned}$$

We obtain:

$$\begin{aligned} &\varphi(hg_1, g_2) + \varphi(hg_1g_2, g_3) - \varphi(hg_1, g_2g_3) \\ &= \varphi(hh_1\gamma_{(1)}, h_2\gamma_{(2)}) + \varphi(hh_1\gamma_{(1)}h_2\gamma_{(2)}, h_3\gamma_{(3)}) - \varphi(hh_1\gamma_{(1)}, h_2\gamma_{(2)}h_3\gamma_{(3)}) \\ &= \varphi(hh_1\gamma_{(1)}, h_2\gamma_{(2)}) + \varphi(hh_1h_4\gamma_{(12)}, h_3\gamma_{(3)}) - \varphi(hh_1\gamma_{(1)}, h_2h_5\gamma_{(23)}) \\ &\stackrel{(9)}{=} \theta_{(1)}(h_2\gamma_{(2)}) + \varphi(hh_1, \gamma_{(1)}h_2\gamma_{(2)}) - \varphi(hh_1, \gamma_{(1)}) + \theta_{(12)}(h_3\gamma_{(3)}) \\ &\quad + \varphi(hh_1h_4, \gamma_{(12)}h_3\gamma_{(3)}) - \varphi(hh_1h_4, \gamma_{(12)}) - \theta_{(1)}(h_2h_5\gamma_{(23)}) \\ &\quad - \varphi(hh_1, \gamma_{(1)}h_2h_5\gamma_{(23)}) + \varphi(hh_1, \gamma_{(1)}) \\ &= \theta_{(1)}(h_2\gamma_{(2)}) + \theta_{(12)}(h_3\gamma_{(3)}) - \theta_{(1)}(h_2h_5\gamma_{(23)}) + \varphi(hh_1, h_4\gamma_{(12)}) \\ &\quad - \varphi(hh_1h_4, \gamma_{(12)}) + \varphi(hh_1h_4, h_4^{-1}h_6\gamma_{(123)}) - \varphi(hh_1, h_6\gamma_{(123)}) \\ &\stackrel{(*)}{=} \theta_{(1)}(h_2\gamma_{(2)}) + \theta_{(12)}(h_3\gamma_{(3)}) - \theta_{(1)}(h_2h_5\gamma_{(23)}) + \varphi(hh_1, h_4) - \varphi(h_4, \gamma_{(12)}) \\ &\quad + \varphi(hh_1h_4, h_4^{-1}) - \varphi(h_4^{-1}, h_6\gamma_{(123)}) \\ &\stackrel{(o)}{=} \theta_{(1)}(h_2\gamma_{(2)}) + \theta_{(12)}(h_3\gamma_{(3)}) - \theta_{(1)}(h_2h_5\gamma_{(23)}) - \varphi(h_4, \gamma_{(12)}) \\ &\quad - \varphi(h_4^{-1}, h_6\gamma_{(123)}) + \varphi(h_4, h_4^{-1}) + \varphi(hh_1, 1) \end{aligned}$$

which is independent of  $h$  since  $\varphi(hh_1, 1) = \varphi(1, 1)$ . Applications of (7) yield (\*) and (o). For (\*) we applied (7) twice, namely for  $(h_1, g_2, g_3) = (hh_1, h_4, \gamma_{(12)})$  and for  $(h_1, g_2, g_3) = (hh_1h_4, h_4^{-1}, h_6\gamma_{(123)})$ . For (o) we used  $(h_1, g_2, g_3) = (hh_1, h_4, h_4^{-1})$ . (Actually in the last case (7) is the cocycle equation on  $H$ .)  $\square$

In Corollary 4 we have described the general solution of (7) under the assumption that we know all solutions of the cocycle equation on  $H$ . Now we are able to prove the main theorem. It gives the desired answer in terms of solutions of the cocycle equation on  $H$ .

**Theorem 6.** *Consider a transitive action of a group  $(G, \cdot)$  on a set  $X$  and let  $(K, +)$  be an abelian group. Let  $x_0 \in X$  and let  $H$  denote the stabilizer of  $x_0$ . The general solution  $f: G \times G \times X \rightarrow K$  of (1) is given by (6) where  $\varphi: G \times G \rightarrow K$  is a solution of (7).*

*Proof.* If  $f$  satisfies (1), then we have already shown that  $f$  has the desired representation.

Assume that  $f$  is given by (6) where  $\varphi$ , described in Corollary 4, is a solution of (7). According to Lemma 5,  $f$  is well defined. We still have to prove that  $f$  satisfies (1). Consider some  $x \in X$ , then there exists some  $g \in G$  so that  $x = g^{-1}x_0$ . Simple calculations show that

$$\begin{aligned} & f(g_1, g_2, x) + f(g_1g_2, g_3, x) \\ &= f(g_1, g_2, g^{-1}x_0) + f(g_1g_2, g_3, g^{-1}x_0) \\ &= \varphi(g, g_1) + \varphi(gg_1, g_2) + \varphi(gg_1g_2, g_3) - \varphi(g, g_1g_2g_3) \end{aligned}$$

and also

$$\begin{aligned} & f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x) \\ &= f(g_2, g_3, g_1^{-1}g^{-1}x_0) + f(g_1, g_2g_3, g^{-1}x_0) \\ &= \varphi(gg_1, g_2) + \varphi(gg_1g_2, g_3) + \varphi(g, g_1) - \varphi(g, g_1g_2g_3). \quad \square \end{aligned}$$

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