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Study of Self Organized Criticality in Spiral Sandpile Model

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Abstract

Bak, Tang and Wiessenfeld's (BTW) Sandpile model [1] under a new constraint is studied on square lattice in two dimensions. Some non trivial properties of the avalanche clusters are observed. The abelian properties of the model are studied. The new model is found to bear the Self organized Criticality characteristics. The critical exponents are calculated which are found to be different from all the other sandpile models. Thus it is concluded that this model defines a new universality class with it's new critical exponents.

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Certificate

This is to certify that the work contained in this project report entitled **'Study of Self Organized Criticality in Spiral Sandpile Model'** prepared by Debabrata Deb, a student of M.Sc. 4th Semester, Department of Physics, IIT Guwahati was carried out under my supervision.

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Acknowledgment

Here I would like to remember those unforgettable efforts of my respected guide Dr. S. B. Santra to help and encourage me during the whole project period. There are many instances when he gave me lots of time and efforts without being annoyed by my silly mistakes and pranks.

My parents have always been the main source of my self-confidence during this project.

It would be very unlikely if I do not mention the name of Santanu Sinha who had helped me in innumerable ways to face and solve many problems and above all to get the feeling of doing research.

The friends and classmates have always been supporting in my all good and bad times.

Debabrata Deb

1 Introduction

In 1987, Bak, Tang and Wiessenfeld (BTW) have shown that there is a class of systems, driven by external agencies, naturally evolve into a state of no single characteristic size or time. The phenomenon is termed as Self Organized Criticality (SOC)[1, 2, 3]. BTW used the example of sandpile to illustrate the idea of SOC. If a sandpile is build on a horizontal base by slowly adding sand grains one after another (external drive) then after a long time the pile reaches a steady state where it's surface makes a constant angle with the horizontal. At this state addition of each sand makes the pile critical, which results to a burst of activity on the surface of the pile and during which some of the sand grains are thrown out of the pile. BTW called this as an *avalanche*. When avalanche is stopped, the pile becomes under critical again. In this way at the steady state the total in flow and out flow of sand mass remain constant. Thus the dynamics of a sandpile has all the ingredients of SOC, namely the slow external drive and a steady state, and hence BTW argued that avalanches in a model sandpile will have a power law distribution in the steady state. In the first part of this project, a detailed discussion of SOC and BTW sandpile model was given. BTW model was simulated on a two dimensional square lattice and it's results were verified.

After the remarkable work of BTW, there has been an active research on the sandpile model, as a result at present many different versions of SPM are available. In all these models, variation of the dynamical rule of the original BTW model was made, in order to check that whether the system still bears the SOC characteristics or not. Further more, if it does so, then is there any change to the critical exponents corresponding to these changes in the dynamical rules. Like the BTW model, the dynamical rules in different models are defined in such a way that they depict some physical situation. Such as in Manna stochastic sandpile model the sand grains in an avalanche moves in different direction with some probability, which may be taken as a simple phenomenological attempt to take into account, the variation in shape and smoothness of different grains in real granular media [5, 6, 14]. In the Dhar directed sandpile model the sand grains moves along a fixed direction, which may be thought to take into account a physical situation where there exists a preferred direction (that of the steepest descent, say) in the sandpile [7].

In this part of the project, a new variant of BTW model is considered where the sandpile is being made on a horizontal disk and the disk is under a constant slow rotation about an axis passing through it's center and perpendicular to it's plane. Now it can be understood that due to this slow rotation of the disk the sand grain on the surface of the sandpile will have a spiraling effect. Being motivated by this physical consideration a new constraint is applied to the BTW model. As the sand grains will be following a spiraling path model is named as *Spiral Sandpile Model* (SSM). It is found that the model shows some nontrivial behavior in its dynamics and the flow of the sand grains in an avalanche is very complicated. However it is still found that the model bears the SOC characteristics but the critical exponents are different from the other models. Thus this Spiral model defines a new universality class with it's new critical exponent values.

2 The Spiral Sandpile Model

Here SSM is defined on a 2D square lattice as: each lattice site is assigned an integer z_i, called the height of the site. At the beginning all the z_i are set to zero and then at each time step a sand grain is added to the randomly chosen lattice site z_i. A site is called active when the height of a site becomes greater than or equal to the threshold value $z_c = 2$ and an active site topples by following the toppling rule as:

$$z_i \rightarrow z_i - 2 \tag{2.1}$$

$$z_j \rightarrow z_j + 1 \tag{2.2}$$

where the height of the toppled site is reduced by 2 and two of the four neighboring sites $\{j = 1, 2, 3, 4\}$ gain one unit of sand grain each. Now for the central site where the avalanche is started the selection of the two neighboring site is accomplished stochastically as done by Manna [5]. To do the stochastical selection of a site, a random number r, say, is invoked. The random number gives uniformly distributed values between zero to one. These values are divided into four equal quarters. Now on observing the value of the random number r we choose the site *1* if r falls into first quarter or *2* if r falls into second quarter and so on. But if this toppling of the central site makes any neighboring sites active then in that case the toppling is accomplished not stochastically but deterministically (as in BTW model) by choosing two neighboring sites as: one from the forward direction and another by giving a clockwise rotation to the first direction. On saying forward direction it is meant that the direction from which the site has received the last sand. Now to keep a track of the direction of the last sand came each site is assigned another integer d_i. The integer d_i can take values from one to four depending upon the site from where the last sand has came (figure 1). Thus the rule of selecting the neighbors for a non central site is:

$$d_j = d_i \tag{2.3}$$

$$d_j = d_i + 1 \tag{2.4}$$

The figure 1. shows a situation where the central crossed marked site was active first, which has toppled by choosing randomly the sites numbered with one and three. As result these two sites also has become active and then they topples by following the SSM toppling rule. Thus in SSM the toppling rule is a mixture of Manna and BTW models in addition to the spiraling constraint.

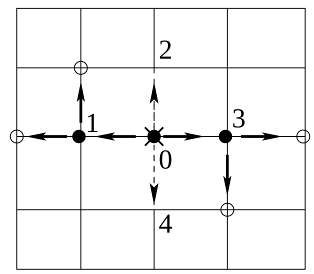

Figure 1: Depicting the toppling of the active sites following the SSM toppling rule. The avalanche has stated from the central crossed marked site.

Now as it was mentioned earlier that SSM shows some nontrivial features here some of those non-triviality are discussed. The figures 2 (a),(b)and (c) shows three pictures of a single avalanche cluster obtained on a 128×128 square lattice. In the figure each site is given a particular color depending upon the number of times it has toppled. Also in all the figures the site from where the avalanche has started is indicated by a solid large spot of pink color. In the figure 2, the red color is given to the site which has toppled maximum and then the color is changed continuously from red to blue through green as the toppling number decreases to a minimum of one. Now it can be observed that there is total mixing of the color due to which there are small small islands of different colors. This mixing of colors indicate that during the avalanche the there has been some regions around which the sand grains has spiraled making some zones where the sites has toppled a large number of times. This is totally a new feature of SSM which was absent in all other previously studied sandpile models where the avalanche wave starts from the central site and then decays away to the boundary. It can also be observed that there are some holes inside the avalanche cluster, which also reveals that there are some sites within the avalanche cluster which has never been toppled during the avalanche. The presence of wholes also makes the SSM different from the other sandpile models as the avalanche in other models are always compact. It is argued that nonabelian sandpile model the avalanche clusters cannot have holes [8, 9], so the presence of holes in the clusters indicates that the SSM will be non-abelian. The abelian property will be

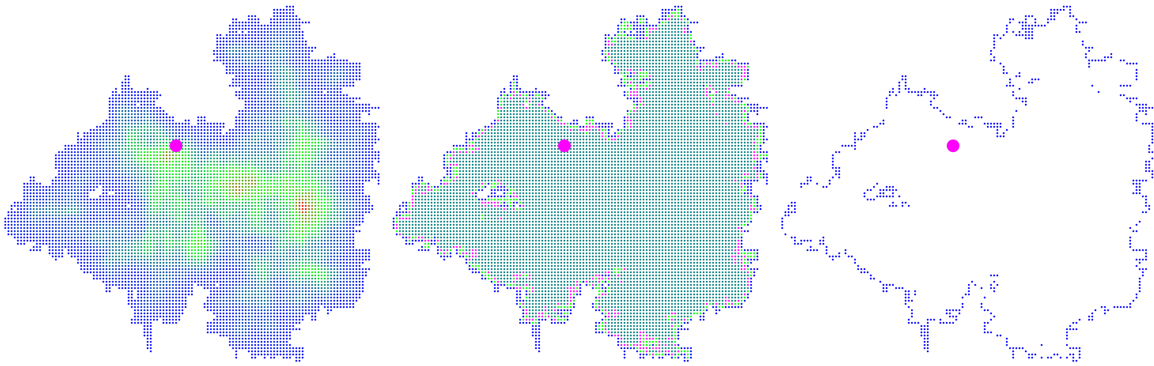

Figure 2: All three pictures are of same avalanche cluster generated on a 128×128 square lattice. 2 (a) Color codes given to the sites depending upon the number of times a particular site has toppled. Red color is given to the site which has toppled maximum and then the color is changed continuously from red to blue as the toppling number is decreased to a minimum of one. 2 (b) Only the sites which has toppled once, twice, thrice and four times are shown by giving the different colors. 2 (c) The sites that has toppled only once are shown.

discussed in the next section. In the figure 3. only the sites toppled once, twice, thrice and four times are shown with respective colors as blue, green, read and dark green. The picture reveals that there does not exist any definite continuous contour of the sites that has toppled same time, rather they are dis-joined. In the figure 3 only the sites which has toppled once are shown, which again confirms the above fact. This overlapping of the regions with sites toppled different number of times is special property which was also absent in the other models.

Now with all this new features, it becomes quite interesting to find out whether the SSM still bears the SOC characteristics or not. In the following sections, first the abelian property of SSM will be checked and then the some scaling relations will be derived. Finally the simulation results of SSM will be given.

3 Verification of the Abelian Property

A sandpile model is said to be Abelian when the final stable configuration of the system is independent of the sequence in which the sand grains are added to the system to reach a stable configuration [12]. BTW, Dhar and Manna sandpile models are found to be abelian [1],[7] [5] [9],[10], [14]. But the spiral model is found to be non-abelian in nature. This can be realized from the figure 5., where the spiral model toppling rules are applied in a 2×2 cell by adding two sand grains in two different sequences. The stable configurations found in the two sequences in figure 3(a) and (b) are not same. But still there is an open opportunity for proving this non-abelian property in more rigorous way by following some operator algebra as done in case of other sandpile models [14].

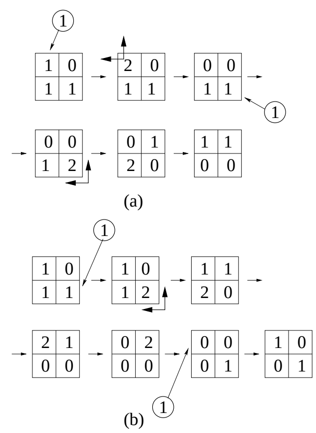

Figure 3: Non-abelian Property of Spiral Model

4 Scaling relations

The measurement of the avalanche properties in a sandpile model can only be done after the system has reached a steady state, by steady state it is meant that where the total influx and out flux of sand grains in the system are same i.e. at the steady state the average height of the lattice column will fluctuate about an average value. Thus the characterization of the steady state can be done by measuring the average height of the sandpile at each time step (addition of each sand grain). Now at a time t, the average height $\langle z \rangle$ can be defined as:

$$\langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i \tag{4.5}$$

where the N is the total number of lattice site. Now after the system has reached the steady state, the three quantities which mainly characterizes an avalanche are size (s): the total no of distinct sites toppled during the avalanche, time (t): the total number of simultaneous updates of the whole lattice that must be executed in order to make all the sites under critical and length (l): the linear extent of the avalanche, which can be defined as:

$$l = \frac{1}{|A|}\sum_{r\epsilon A}|\mathbf{r} - \mathbf{R}_{cm}| \tag{4.6}$$

where A is the set of all toppled sites and $\mathbf{r}$ is distance of the toppled site from the center of mass of the avalanche defined as:

$$\mathbf{R}_{cm} = \frac{1}{|A|}\sum_{r\epsilon A}\mathbf{r} \tag{4.7}$$

Also at the steady state the three probability distribution functions corresponding to these three quantities s, t, l can be defined as: $P(s)$ - Probability that an avalanche of size s irrespective of the avalanche's life time and length will occur, $P(t)$ - Probability that an avalanche of life time t irrespective of the avalanche's length and size will occur and $P(l)$ - Probability that an avalanche of length l irrespective of the avalanche's life time and size will occur.

Now if the SSM has to show the SOC characteristics then at the steady state all the three probabilities should have power law distribution as:

$$P(s) \sim s^{-\tau_s}, \qquad P(t) \sim t^{-\tau_t}, \qquad P(l) \sim l^{-\tau_l} \tag{4.8}$$

As s, t and l are the three different measurements of the same random avalanche cluster, they are necessarily dependent variables. We assume that these quantities are related to one another by power laws as:

$$E[S|T=t] \sim t^{\gamma_{ts}}, \qquad E[S|L=l] \sim l^{\gamma_{ls}}, \qquad E[T|L=l] \sim l^{\gamma_{lt}} \tag{4.9}$$

such that the exponents satisfy the relation

$$\gamma_{ls} = \gamma_{ts}\gamma_{lt} \tag{4.10}$$

A set of scaling relations between the critical exponents can be obtained from the following identity:

$$\int dyE[X|Y=y]P(Y=y) = \int dzE[X|Z=z]P(Z=z) \tag{4.11}$$

which is fulfilled for any set of three stochastic variables X,Y,Z.
Now for $X=S$, $Y=T$, $Z=L$ and assuming that $L = T^{1/\gamma_{lt}}$, then

$$\begin{aligned}
\int E[S|T=t]P(T=t)dt &= \int E[S|L=l]P(L=l)dl \\
or, \qquad \int t^{\gamma_{ts}-\tau_t}dt &= \frac{1}{\gamma_{lt}}\int t^{\frac{-\tau_l+\gamma_{ls}+1}{\gamma_{lt}}-1}dt \\
or, \qquad \gamma_{ts}-\tau_t &= \frac{-\tau_l+\gamma_{ls}+1}{\gamma_{lt}}-1 \\
or, \qquad \tau_t &= 1+\gamma_{ts}+\frac{\tau_l-\gamma_{ls}-1}{\gamma_{lt}}
\end{aligned} \tag{4.12}$$

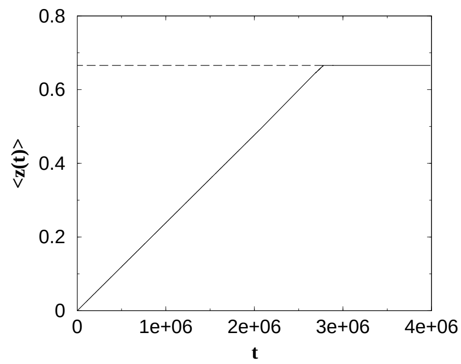

Figure 4: Plot of $\langle z \rangle$ against time t for a system of size 2048×2048

Now for $X = T$, $Y = S$, $Z = L$ and assuming that $L = S^{1/\gamma_{ls}}$, then

$$\tau_s = 1 + \frac{1}{\gamma_{ts}} + \frac{\tau_l - \gamma_{lt} - 1}{\gamma_{ls}} \tag{4.13}$$

and finally for $X = L$, $Y = S$, $Z = T$ and assuming that $L = S^{1/\gamma_{ls}}$, then

$$\tau_s = 1 + \frac{1}{\gamma_{ls}} + \frac{(\tau_t - 2)\gamma_{lt} - 1}{\gamma_{ts}\gamma_{lt}} \tag{4.14}$$

Since $\gamma_{ls} = \gamma_{ts}\gamma_{lt}$, the relations (4.12), (4.13) and (4.14) give rise to the scaling relations as: exponents as :

$$\gamma_{ts} = \frac{(\tau_t - 1)}{(\tau_s - 1)}, \qquad \gamma_{ls} = \frac{(\tau_l - 1)}{(\tau_s - 1)}, \qquad \gamma_{lt} = \frac{(\tau_l - 1)}{(\tau_t - 1)}, \tag{4.15}$$

In the next section all the six critical exponents will be calculated explicitly and the above scaling relations will be proved.

5 Results and discussions

Simulation has been done on a square lattice of size 2048×2048. The figure 4. shows the average height of the sandpile. The average height first increases linearly from zero to a certain value and then fluctuates around an average value. The is found to equal to 0.66. This can be explained as: at each time step all the sites are under critical, so the height of all the sites could be either one or zero. So on the average the total height would be between zero and one. The calculated value of the average height shows that on the average there are 66% of the lattice sites are of height one.

Now after the sandpile has reached the steady state the measurement of the different avalanche properties are started and for that a total of 5×10^4 samples are taken. Then the measured values are averaged over these samples.

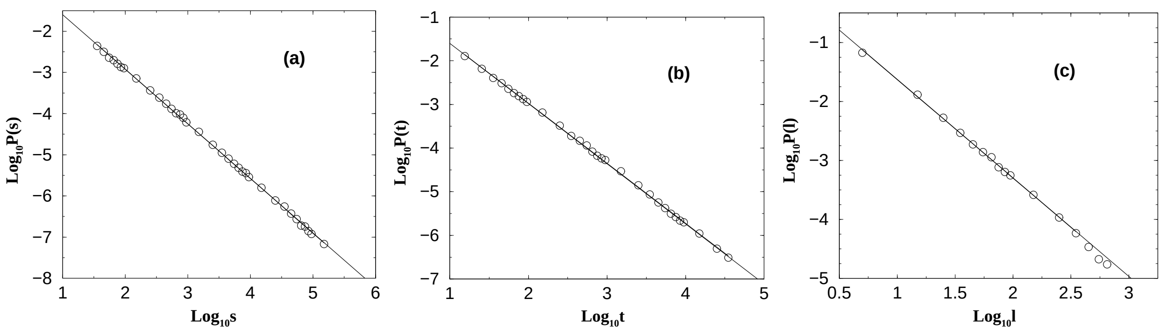

Figure 5: The avalanche probability distributions. (a) size, (b) time, (c) length

The figures 5 (a),(b) and (c) shows the probability distribution plots of s, t and l respectively. From these plots, it can be observe that probability distribution of s, t and l indeed follow power law behavior. Here another point can also be observed that the power law behavior of s and t are very excellent while that of length is not. This can be explained by taking into account that s and t can have values up to the order of the whole lattice size, say 2048×2048 in this case, while l can take only values up to the linear extent of the lattice i.e of the order of 2048 here. The critical exponent values of the probability distributions are calculate by measuring the slope of the fitted lines and the values are found to be $\tau_s = \frac{4}{3}$, $\tau_t = \frac{25}{18}$ and $\tau_l = \frac{5}{3}$. The figures 6 (a),(b) and (c) shows the plots of s versus t, s versus l and t versus l respectively in log-log scale. As the s, t and l are interrelated by the power laws defined in equation 4.9, so the slope of these plots are the critical exponents. The measurement of the slope is done by finding the best fitted straight line which passes through the origin (0, 0) as wel as the measured data points. This is done because the point (0, 0) is the most trivial point in all the plots as when any one of s, t or l is zero then others should also be zero, for example, if time $t = 0$ then the size or length should be equal to 0. But here all the considered avalanche samples are of size greater than zero so in the measured data, point (0, 0) is absent. From such best fitted straight lines the estimated values of the exponents are $\gamma_{ts} = \frac{7}{6}$, $\gamma_{ls} = 2$ and $\gamma_{lt} = \frac{12}{7}$. It is to be noted that the value of γ_{ls} is estimated as 2 which is quite obvious and proves the correctness of the measurement. Because the given definition of size (s) of an avalanche also represents the total lattice points within the avalanche cluster which is nothing but the total area of the avalanche. As one know that in a euclidean two dimensional space the area (size) goes as the square of its linear extent. Now having all these values of critical exponents in hand it can be checked that whether they satisfy the scaling relations 4.10 and 4.15 or not. The scaling relation 4.10 can be verified by using

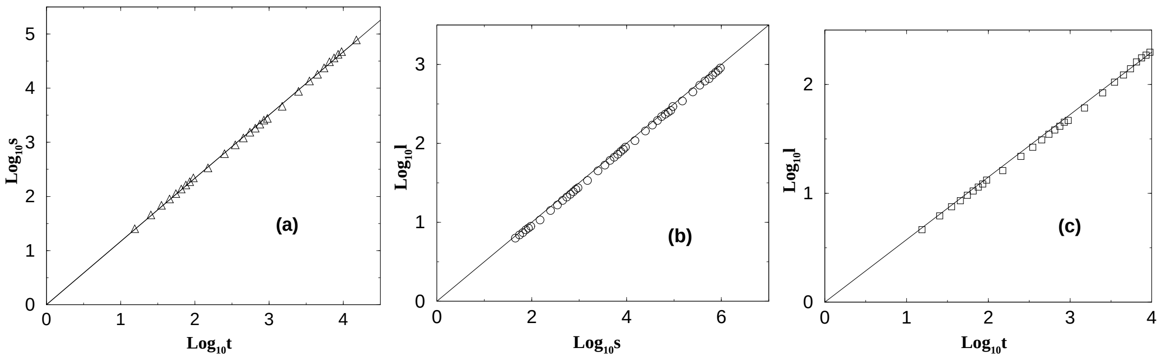

Figure 6: The inter dependence of avalanche size (s), time (t) and length (l). (a) Variation of size with time. (b) Variation of length with size. (c) Variation of length with time

the values of γ_{ts} and γ_{lt} to find the value of γ_{ls} as:

$$\gamma_{ls} = \gamma_{ts}\gamma_{lt} = \frac{7}{6} \times \frac{12}{7} = 2 \tag{5.16}$$

which is exactly equal to the measured value of γ_{ls}. Also the relation 4.15 can be verified by finding the values of γ_{ts}, γ_{ls} and γ_{lt} using the values of τ_s, τ_t and τ_l as:

$$\gamma_{ts} = \frac{(\tau_t - 1)}{(\tau_s - 1)} = \frac{\frac{25}{18} - 1}{\frac{4}{3} - 1} = \frac{\frac{7}{18}}{\frac{1}{3}} = \frac{7}{6} \tag{5.17}$$

$$\gamma_{ls} = \frac{(\tau_l - 1)}{(\tau_s - 1)} = \frac{\frac{5}{3} - 1}{\frac{4}{3} - 1} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2 \tag{5.18}$$

$$\gamma_{lt} = \frac{(\tau_l - 1)}{(\tau_t - 1)} = \frac{\frac{5}{3} - 1}{\frac{25}{18} - 1} = \frac{\frac{2}{3}}{\frac{7}{18}} = \frac{12}{7} \tag{5.19}$$

Thus the critical exponents τ_s, τ_t, τ_l and γ_{ts}, γ_{ls}, γ_{lt} satisfy the scaling relation 4.10 exactly.

In principle the exact power law behavior of the sandpile at the steady state would be observed at the limit when lattice size $L \to \infty$ [2, 3]. So to confirm that the measured critical exponents are not effected by the finite size of the lattice used in the simulation the plot of the the probability distribution exponent of size, τ_s is plotted against $\frac{1}{L}$ in figure 7 for L = 128, 256, 512, 1024 and 2048. From the plot it can be seen that the exponent approaches to the exact value $\frac{4}{3}$ as $L \to \infty$. So it can be concluded that the estimated value of τ_s is free of finite lattice size effects and the value of $\tau_s = \frac{4}{3}$ is exact and as the exponents are related by the scaling relations so it can be said that the other critical exponents are also exact.

Now the results of SSM can be compared with some of the other models. In the table 1. the critical exponents of BTW, Dhar, Manna, and SSM model are given. Here

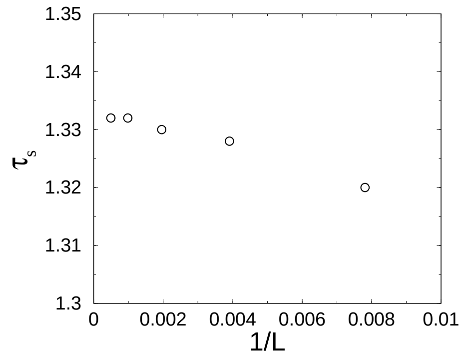

Figure 7: Variation of τ_s with lattice size

one point is be made clear that the data for BTW and Manna model shows that they belong to different Universality class, but there is big controversy about this [15, 16, 17, 18, 19, 20, 21, 22, 21]. So keeping this in mind, here only the most convincing data are shown. The table shows that the exponent γ_{ls} is equal to 2 for all the model. This is expected because as we discussed earlier that size should go quadratically with length in an euclidean two dimensional space. The table shows that the probability distribution critical exponents size and length in SSM and BTW model are same but that of time is different. This can be accounted by considering the fact that in SSM the avalanche waves generally have a spiraling effect around some regions within the avalanche cluster so it takes a longer amount of time for an avalanche to die away than in BTW. In this way the rotational constraint makes SSM different from BTW. Thus comparing the data with other models it can be concluded that the SSM belongs to a new universality class with its new set of critical exponents.

Table 1: Critical Exponents of different Sandpile models

$Model$	τ_s	τ_t	τ_l	γ_{ts}	γ_{ls}	γ_{lt}
BTW[1, 19]	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$	2	$\frac{4}{3}$
Dhar[10]	$\frac{5}{4}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{8}{5}$	2	$\frac{5}{4}$
Manna[5, 18, 19]	$\frac{11}{8}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{4}{3}$	2	$\frac{3}{2}$
SSM	$\frac{4}{3}$	$\frac{25}{18}$	$\frac{5}{3}$	$\frac{7}{6}$	2	$\frac{12}{7}$

6 Conclusion

The original BTW sandpile model under a new rotational constraint is studied. The model is given a name Spiral Sandpile Model (SSM). The simulation is done on two dimensional square lattice. Some non-trivial avalanche properties are observed such as the avalanche clusters have some small holes in it and the contour of the sites toppled different times are observed to overlap. This properties are new and exclusive to SSM only. Also the avalanche dynamics is found to be non-abelian. Even though SSM have all these new exciting properties in its dynamics, it is still found that SSM bears the SOC characteristics. The critical exponents are calculated as $\tau_s = \frac{4}{3}$, $\tau_t = \frac{25}{18}$, $\tau_l = \frac{5}{3}$, $\gamma_{ts} = \frac{7}{6}$, $\gamma_{ls} = 2$ and $\gamma_{lt} = \frac{12}{7}$. Finally these critical exponents are compared with some of the well known variants of BTW sandpile model. From this comparisons it is then concluded that SSM belongs to a new universality class with its new critical exponent values. Thus this new model of sandpile again throws up many questions such as why the universality has broken up due to the change in the dynamical rule in micro level, even if it is known that the critical phenomena is a macroscopic behavioral out come of a system at critical state? How the abelian property of a system is related with the universality class? In the near future the results of this new sandpile model and the answer of these questions may open up many new door to study other self organized critical systems.

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