Approximate convexity and Korovkin type theorems

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Approximation by positive operators I.

The Korovkin Approximation Theorem
Let $T_n : C([0, 1]) \to C([0, 1])$ be a sequence of positive linear operators such that 
$$\|T_n p_i - p_i\| \to 0 \quad (i = 0, 1, 2),$$
where $p_i(x) := x^i$. Then 
$$\|T_n f - f\| \to 0 \quad (f \in C([0, 1])).$$

Corollary
Let $T : C([0, 1]) \to C([0, 1])$ be a positive linear operator such that 
$$Tp_i = p_i \quad (i = 0, 1, 2).$$
Then 
$$Tf = f \quad (f \in C([0, 1])).$$
Approximation by positive operators II.

A Generalized Korovkin Theorem

Let \( T_n : C([0, 1]) \to C([0, 1]) \) be a sequence of positive linear operators such that

\[
\| T_n p_i - p_i \| \to 0 \quad (i = 0, 1) \quad \text{and} \quad \| T_n g - g \| \to 0,
\]

where \( g \in C([0, 1]) \) is a strictly convex function. Then

\[
\| T_n f - f \| \to 0 \quad (f \in C([0, 1])).
\]

Corollary

Let \( T : C([0, 1]) \to C([0, 1]) \) be a positive linear operator such that

\[
T p_i = p_i \quad (i = 0, 1) \quad \text{and} \quad T g = g,
\]

where \( g \in C([0, 1]) \) is a strictly convex function. Then

\[
T f = f \quad (f \in C([0, 1])).
\]
An Approximation Problem

Let $T_n : C([0, 1]) \to C([0, 1])$ a sequence of positive linear operators such that

$$\|T_n p_i - p_i\| \to 0 \quad (i = 0, 1) \quad \text{and} \quad \|T_n g - h\| \to 0,$$

where $g, h \in C([0, 1])$ given functions. Then what can be expected about the limit of the sequences

$$T_n f \quad (f \in C([0, 1]))?$$

Subproblem

Let $T : C([0, 1]) \to C([0, 1])$ be a positive linear operator such that

$$Tp_i = p_i \quad (i = 0, 1) \quad \text{and} \quad Tg = h,$$

where $g, h \in C([0, 1])$ are given functions. Then what can be expected about $T$?
Positive Operators and Convexity

**Theorem**

Let $T : C([0, 1]) \rightarrow C([0, 1])$ be a positive linear operator such that $Tp_i = p_i \ (i = 0, 1)$. Then, for every convex $g \in C([0, 1])$, 

$$g \leq Tg \leq g(0)p_0 + (g(1) - g(0))p_1.$$ 

In particular, 

$$p_2 \leq Tp_2 \leq p_1.$$ 

**Proof**

By the convexity of $g$ and the positivity of $T$, 

$$g \leq g(0)p_0 + (g(1) - g(0))p_1 \quad \Rightarrow \quad Tg \leq g(0)p_0 + (g(1) - g(0))p_1.$$ 

On the other hand, for every $x \in ]0, 1[$ there exists $c \in \mathbb{R}$ such that 

$$(g(x) - cx)p_0 + cp_1 \leq g \quad \Rightarrow \quad (g(x) - cx)p_0 + cp_1 \leq Tg \quad \Rightarrow \quad g(x) \leq (Tg)(x).$$
A Korovkin-type Theorem

Let $T_n : C([0, 1]) \to C([0, 1])$ be a sequence of positive linear operators such that

$$T_n p_i \to p_i \quad (i = 0, 1) \quad \text{és} \quad T_n g \to g(0)p_0 + (g(1) - g(0))p_1,$$

where $g \in C([0, 1])$ is a strictly convex function. Then

$$T_n f \to f(0)p_0 + (f(1) - f(0))p_1 \quad (f \in C([0, 1])).$$

Corollary

Let $T : C([0, 1]) \to C([0, 1])$ be a positive linear operator such that

$$Tp_i = p_i \quad (i = 0, 1) \quad \text{and} \quad Tg = g(0)p_0 + (g(1) - g(0))p_1,$$

where $g \in C([0, 1])$ is a strictly convex function. Then

$$Tf = f(0)p_0 + (f(1) - f(0))p_1 \quad (f \in C([0, 1])).$$
Approximation by positive operators V.

Sketch of The Proof

Without loss of generality, we may assume that \( g(0) = g(1) = 0 \).
Let \( f \in C([0, 1]) \). For every \( \varepsilon > 0 \),
\[
\sup_{0 < x < 1} \left| \frac{f(x) - (f(0) + (f(1) - f(0))x)}{-g(x)} \right| - \varepsilon < \infty.
\]

Thus, for all \( \varepsilon > 0 \), there exists \( K > 0 \) such that
\[
|f - (f(0)p_0 + (f(1) - f(0))p_1)| \leq \varepsilon + K(-g).
\]

Hence
\[
|T_n[f - (f(0)p_0 + (f(1) - f(0))p_1)]| \leq T_n|f - (f(0)p_0 + (f(1) - f(0))p_1)|
\leq \varepsilon + KT_n(-g).
\]

Therefore
\[
\limsup_{n \to \infty} \| T_n[f - (f(0)p_0 + (f(1) - f(0))p_1)] \| \leq \varepsilon.
\]
Theorem
Let \( T : C([0, 1]) \to C([0, 1]) \) be a positive linear operator such that \( Tp_0 = p_0 \) and \( Tp_1 = p_1 \). Then the following are equivalent:

(i) \( T^n f \to f(0)p_0 + (f(1) - f(0))p_1 \) (\( f \in C([0, 1]) \)).

(ii) \( T^n g \to g(0)p_0 + (g(1) - g(0))p_1 \) for some strictly convex function \( g \in C([0, 1]) \).

(iii) Every fixed point of \( T \) is of the form \( ap_0 + bp_1 \).

Corollary
Let \( T : C([0, 1]) \to C([0, 1]) \) be a positive linear operator such that \( Tp_0 = p_0 \) and \( Tp_1 = p_1 \). Assume that there exists a strictly concave function \( g \in C([0, 1]) \) such that \( g(0) = g(1) = 0 \) and \( Tg \leq \gamma g \) for some \( 0 \leq \gamma < 1 \). Then

\[
T^n f \to f(0)p_0 + (f(1) - f(0))p_1 \quad (f \in C([0, 1])).
\]
Averaging operators I.

**Definition**

Let $\mu$ be a Borel regular probability measure on $[0, 1]$ with a non-singleton support. Denote by $\mu_1$ the first moment of $\mu$ and define the **averaging operator** $T_\mu : C([0, 1]) \rightarrow C([0, 1])$ by

$$(T_\mu f)(s) := \begin{cases} 
\int_{[0,1]} f\left(\frac{st}{\mu_1}\right) d\mu(t) & \text{if } s \in [0, \mu_1], \\
\int_{[0,1]} f\left(\frac{s+t-st-\mu_1}{1-\mu_1}\right) d\mu(t) & \text{if } s \in [\mu_1, 1].
\end{cases}$$

**Theorem**

Under the above assumptions, $T_\mu : C([0, 1]) \rightarrow C([0, 1])$ is a positive linear operator such that $T_\mu p_0 = p_0$ and $T_\mu p_1 = p_1$. Moreover, all fixed points of $T_\mu$ are of the form $ap_0 + bp_1$ for some $a, b \in \mathbb{R}$. 
Generalized Hermite–Hadamard inequality I.

Theorem

Let $\mu$ be a Borel regular probability measure on $[0, 1]$, with a non-singleton support. Denote by $\mu_1$ the first moment of $\mu$. If $f : I \to \mathbb{R}$ is a convex function, then

$$f(\mu_1 x + (1 - \mu_1)y) \leq \int_{[0,1]} f(tx + (1 - t)y) \, d\mu(t) \quad (x, y \in I).$$

Proof

Let $x, y \in I$, $x \neq y$. Then, by the convexity of $f$, there exists $c \in \mathbb{R}$, such that

$$f(\mu_1 x + (1 - \mu_1)y) + c(t - \mu_1) \leq f(tx + (1 - t)y) \quad (t \in [0, 1]).$$

Integrating this inequality by the variable $t$ with respect to the measure $\mu$, the statement follows.
Generalized Hermite–Hadamard Inequality II.

The Inverse Problem

Let $\mu$ be a Borel regular probability measure on $[0, 1]$ with a non-singleton support. Let $f : I \to \mathbb{R}$ be a continuous function such that

$$f(\mu_1 x + (1 - \mu_1)y) \leq \int_{[0,1]} f(tx + (1 - t)y) d\mu(t) \quad (x, y \in I).$$

Does this imply the convexity of $f$?
A more general problem

Let \( \mu \) be a Borel regular probability measure on \([0, 1]\) with a non-singleton support. Let \( \varepsilon : I \times I \to \mathbb{R} \) be an error function such that \( \varepsilon(x, x) = 0 \) for all \( x \in I \) and let \( f : I \to \mathbb{R} \) be a continuous function such that

\[
f(\mu_1 x + (1 - \mu_1) y) \leq \int_{[0,1]} f(tx + (1 - t)y) d\mu(t) + \varepsilon(x, y) \quad (x, y \in I).
\]

What can be said about \( f \)? Is \( f \) approximatively convex in some sense?

Contributions to Approximate Convexity

Theorem

Let \( \mu \) be a Borel regular probability measure on \([0, 1]\) with a non-singleton support. Let \( \varepsilon : I^2 \rightarrow \mathbb{R} \) such that \( \varepsilon(x, x) = 0 \) for all \( x \in I \) and \( \varepsilon^* : I^2 \times [0, 1] \rightarrow \mathbb{R} \) be a function such that, for all \( x, y \in I \),

\[
\varepsilon^*(x, y, 0) = \varepsilon^*(x, y, 1) = 0 \quad \text{and} \\
\varepsilon^*(x, y, s) \geq \begin{cases} \\
\int_{[0,1]} \varepsilon^*(x, y, \frac{st}{\mu_1}) d\mu(t) + \varepsilon(x, \frac{\mu_1 - s}{\mu_1} x + \frac{s}{\mu_1} y) & s \in [0, \mu_1], \\
\int_{[0,1]} \varepsilon^*(x, y, \frac{t+s-st-\mu_1}{1-\mu_1}) d\mu(t) + \varepsilon(\frac{1-s}{1-\mu_1} x + \frac{s-\mu_1}{1-\mu_1} y, y) & s \in [\mu_1, 1]. \\
\end{cases}
\]

Then every \( f : I \rightarrow \mathbb{R} \) continuous solution of the functional inequality

\[
f(\mu_1 x + (1 - \mu_1) y) \leq \int_{[0,1]} f(t x + (1 - t)y) d\mu(t) + \varepsilon(x, y) \quad (x, y \in I)
\]

also fulfills

\[
f(t x + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon^*(x, y, t) \quad (x, y \in I, t \in [0, 1]).
\]
An Approximative Hermite–Hadamard Inequality III.

Sketch of the proof

Let $x, y \in I$ be fixed and define

$$
\phi(t) := f(tx + (1 - t)y), \quad \psi(t) := \varepsilon^*(x, y, t) \quad (t \in [0, 1]),
$$

$$
e(t) := \begin{cases} 
  \varepsilon(x, \frac{\mu_1-t}{\mu_1} x + \frac{t}{\mu_1} y) & t \in [0, \mu_1], \\
  \varepsilon(\frac{1-t}{1-\mu_1} x + \frac{t-\mu_1}{1-\mu_1} y, y) & s \in [\mu_1, 1]. 
\end{cases}
$$

Then $\psi(0) = \psi(1) = 0$ and $\phi, \psi, e$ are continuous. The conditions of
the theorem yield

$$
\phi - \mathcal{T}_\mu \phi \leq e \leq \psi - \mathcal{T}_\mu \psi.
$$

Iterating this and adding up the inequalities so obtained, we get

$$
\mathcal{T}_\mu^n \phi - \mathcal{T}_\mu^{n+1} \phi \leq \mathcal{T}_\mu^n \psi - \mathcal{T}_\mu^{n+1} \psi \quad \Rightarrow \quad \phi - \mathcal{T}_\mu^{n+1} \phi \leq \psi - \mathcal{T}_\mu^{n+1} \psi.
$$

Now passing the limit $n \to \infty$, the result follows.
Corollary

Let $\mu$ be a Borel regular probability measure on $[0, 1]$ with a non-singleton support. Let $\varepsilon > 0$ and $q \in ]0, 1[$. Then every continuous solution $f : I \rightarrow \mathbb{R}$ of the functional inequality

$$f(\mu_1 x + (1 - \mu_1) y) \leq \int_{[0,1]} f(tx + (1 - t)y) d\mu(t) + \varepsilon |x - y|^q \quad (x, y \in I)$$

also satisfies

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon^* |x - y|^q \min\left(\frac{t^q}{\mu_1^q}, \frac{(1 - t)^q}{(1 - \mu_1)^q}\right)$$

for all $x, y \in I$ and $t \in [0, 1]$, where

$$\varepsilon^* := \frac{\varepsilon}{1 - \max\left(\int_{[0,1]} \frac{t^q}{\mu_1^q} d\mu(t), \int_{[0,1]} \frac{(1 - t)^q}{(1 - \mu_1)^q} d\mu(t)\right)}.$$
We apply the previous Corollary when $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

**Corollary (Házy–Páles [10])**

Let $\varepsilon > 0$ and $q \in ]0, 1[$. Then every continuous solution $f : I \to \mathbb{R}$ of

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + \varepsilon|x-y|^q \quad (x, y \in I)$$

also satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \frac{\varepsilon 2^q |x-y|^q}{1 - 2^q - 1} \min(t^q, (1-t)^q)$$

for all $x, y \in I$ and $t \in [0, 1]$. 
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