GAUSS–COMPOSITION AND INVARIANCE EQUATION FOR TWO-VARIABLE MEANS
1. **Introduction**

Let \( I \subseteq \mathbb{R} \) be a nonvoid open interval. A two-variable continuous function \( M : I^2 \rightarrow I \) is called a *mean* on \( I \) if

\[
\min(x, y) \leq M(x, y) \leq \max(x, y) \quad (x, y \in I)
\]

holds. If both inequalities are strict whenever \( x \neq y \), then \( M \) is called a *strict mean* on \( I \).

A mean \( M \) on \( I \) is said to be *symmetric* if

\[
M(x, y) = M(y, x)
\]

holds for all \( x, y \in I \).

A mean \( M \) on \( \mathbb{R}_+ \) is called *homogeneous* if

\[
M(tx, ty) = tM(x, y)
\]

holds for all \( t, x, y \in \mathbb{R}_+ \).
Classical examples for two-variable symmetric strict and homogeneous means on $\mathbb{R}_+$:
- the arithmetic, the geometric and the harmonic mean:

$$\mathcal{A}(x, y) := \frac{x + y}{2}, \quad \mathcal{G}(x, y) := \sqrt{xy}, \quad \mathcal{H}(x, y) := \frac{2xy}{x + y}.$$

- the power means, also called Hölder means of exponent $p$

$$H_p(x, y) := \begin{cases} 
\left( \frac{x^p + y^p}{2} \right)^{\frac{1}{p}}, & \text{if } p \neq 0, \\
\sqrt{xy}, & \text{if } p = 0.
\end{cases}$$
The two-variable Gini and the Stolarsky means (cf. [7], [11]) are two substantial generalizations of the power means. Given two parameters $p, q \in \mathbb{R}$, the two-variable Gini mean $G_{p,q} : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ is defined, for $x, y \in \mathbb{R}_+$, by

$$G_{p,q}(x, y) = \begin{cases} \left(\frac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}}, & \text{for } p \neq q, \\ \exp\left(\frac{x^p \ln x + y^p \ln y}{x^p + y^p}\right), & \text{for } p = q, \end{cases}$$
and the two-variable Stolarsky mean $S_{p,q} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the following:

$$S_{p,q}(x, y) := \begin{cases} 
\left( \frac{q(x^p - y^p)}{p(x^q - y^q)} \right)^{\frac{1}{p-q}}, & \text{if } (p - q)pq \neq 0, x \neq y, \\
\exp \left( -\frac{1}{p} + \frac{x^p \log x - y^p \log y}{x^p - y^p} \right), & \text{if } p = q \neq 0, x \neq y, \\
\left( \frac{x^p - y^p}{p(\log x - \log y)} \right)^{\frac{1}{p}}, & \text{if } p \neq 0, q = 0, x \neq y, \\
\left( \frac{x^q - y^q}{q(\log x - \log y)} \right)^{\frac{1}{q}}, & \text{if } p = 0, q \neq 0, x \neq y, \\
\sqrt{xy}, & \text{if } p = q = 0, \\
x, & \text{if } x = y.
\end{cases}$$
Other possible generalization:

If $I \subset \mathbb{R}$ is a nonvoid open interval, a two-variable function $M : I^2 \to I$ is called a \textit{quasi-arithmetic mean} on $I$ if there exists a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$ such that

$$M(x, y) = M_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right)$$

for every $x, y \in I$. 


2. **Gauss-iteration, invariance equation**

Given two means $M, N : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $x, y \in \mathbb{R}_+$, the iteration sequence

\[
\begin{align*}
  x_1 &:= x, \\
  y_1 &:= y, \\
  x_{n+1} &:= M(x_n, y_n), \\
  y_{n+1} &:= N(x_n, y_n) \\
  (n &\in \mathbb{N})
\end{align*}
\]

is said to be the *Gauss-iteration* determined by the pair $(M, N)$ with the initial values $(x, y) \in \mathbb{R}^2_+$. It is well-known (cf. [5], [6]) that if $M$ and $N$ are strict means then the sequences $(x_n)$ and $(y_n)$ are convergent and have equal limits $M \otimes N(x, y)$ which is a strict mean of the values $x$ and $y$. The mean $M \otimes N$ defined by this procedure is called the *Gauss composition* of $M$ and $N$.

A very important result ([6]) in characterizing the Gauss composition of means is the following: If $M, N : \mathbb{R}^2_+ \to \mathbb{R}_+$ are two strict means, their Gauss composition $K = M \otimes N$ is the unique strict mean solution $K$ of the functional equation

\[
K(x, y) = K(M(x, y), N(x, y)) \quad (x, y \in \mathbb{R}_+),
\]

which is called the *invariance equation*. 
EXAMPLE. The simplest example when the invariance equation holds is the well-known identity
\[ \sqrt{xy} = \sqrt{\frac{x + y}{2}} \cdot \frac{2xy}{x + y} \quad (x, y \in \mathbb{R}_+), \]
that is,
\[ \mathcal{G}(x, y) = \mathcal{G}(A(x, y), H(x, y)) \quad (x, y \in \mathbb{R}_+), \]
where \( A, \mathcal{G}, \) and \( H \) stand for the two-variable arithmetic, geometric, and harmonic means, respectively.

A less trivial invariance equation is the identity
\[ A \otimes \mathcal{G}(x, y) = A \otimes \mathcal{G}(A(x, y), \mathcal{G}(x, y)) \quad (x, y \in \mathbb{R}_+), \]
where \( A \otimes \mathcal{G} \) denotes Gauss’s arithmetic-geometric mean.
On the history of Gauss's arithmetic-geometric mean

Gauss (1791): \( M(x, y) = A(x, y) \) and \( N(x, y) = G(x, y) \)
- the two sequences became indistinguishable very rapidly!

Gauss (1799): While studying the arc length of the (Bernoulli)-lemniscate he found that the following two values were the same for at least the first 11 digits:

\[
\frac{1}{A \otimes G(1, \sqrt{2})} \quad \text{and} \quad \frac{2}{\pi} \int_0^1 \frac{1}{\sqrt{1 - t^4}} \, dt
\]

He introduced the lemniscate functions and studied the theory of these functions. Later, he examined the elliptic functions (generalizations of the lemniscate functions) and the elliptic integrals and found the general form of \( A \otimes G \), which is

\[
A \otimes G(x, y) = \left( \frac{2}{\pi} \int_0^{\pi/2} \frac{dt}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} \right)^{-1} \quad (x, y \in \mathbb{R}_+).\]
3. Results

Invariance equation for power means
Daróczy and Páles ([6]) obtained the following result:

**Theorem.** For the triplet \((s, p, q) \in \mathbb{R}^3\) the invariance equation

\[ H_s = H_p \otimes H_q \]

holds on the set \(\mathbb{R}_+^2\) if and only if

\((i)\) \(s \neq 0\) then \(p = q = s\);

\((ii)\) \(s = 0\) then \(p + q = 0\).
Invariance equation for quasi-arithmetic means

In 2002, Daróczy and Páles solved the invariance equation for the quasi-arithmetic means:

**Theorem.** The strictly monotone, continuous functions $\varphi$ and $\psi$ satisfy the functional equation

\[
\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y \quad (x, y \in I)
\]

(i.e., $A$ is invariant with respect to the means $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$) if and only if

(i) either there exist constants $p, a, b, c, d$ with $acp \neq 0$ such that

\[
\varphi(x) = ae^{px} + b, \quad \psi(x) = ce^{-px} + d \quad (x \in I);
\]

(ii) or there exist non-zero constants $a, c$ and constants $b, d$ such that

\[
\varphi(x) = ax + b, \quad \psi(x) = cx + d \quad (x \in I).
\]
Furthermore, they solved the invariance equation for quasi-arithmetic means even in the general setting:

**Theorem.** If $\mathcal{M}_\varphi : I^2 \to I$, $\mathcal{M}_\psi : I^2 \to I$ and $\mathcal{M}_\kappa : I^2 \to I$ are quasi-arithmetic means on $I$, then the invariance equation

$$\mathcal{M}_\varphi = \mathcal{M}_\psi \otimes \mathcal{M}_\kappa$$

holds on $I^2$ if and only if there exist a function $f$ which is continuous and strictly monotone on $I$ and a constant $p \in \mathbb{R}$ such that

$$\mathcal{M}_\varphi(x, y) = \mathcal{M}_f(x, y), \quad \mathcal{M}_\psi(x, y) = \mathcal{M}_{\chi_p \circ f}(x, y)$$

and

$$\mathcal{M}_\kappa(x, y) = \mathcal{M}_{\chi_{-p} \circ f}(x, y)$$

hold for every $(x, y) \in I$, where

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0, \\ e^{px} & \text{if } p \neq 0 \end{cases} \quad (x \in I).$$
A generalization of the quasi-arithmetic means

Given two continuous strictly monotone functions \( \varphi_1, \varphi_2 : I \to \mathbb{R} \) such that \( \varphi_1 \) and \( \varphi_2 \) are strictly monotone in the same sense, the generalized quasi-arithmetic mean \( M_\varphi : I^2 \to I \) is defined by

\[
M_\varphi(x, y) := \varphi^{-1}(\varphi_1(x) + \varphi_2(y)) \quad (x, y \in I),
\]

where

\[
\varphi := (\varphi_1, \varphi_2), \quad \varphi := \varphi_1 + \varphi_2.
\]

Our aim is to characterize the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means, that is, to solve the equation

\[
M_\varphi(x, y) + M_\psi(x, y) = x + y \quad (x, y \in I).
\]
This, in detailed form, is equivalent to the functional equation

$$(\varphi_1 + \varphi_2)^{-1}(\varphi_1(x) + \varphi_2(y)) + (\psi_1 + \psi_2)^{-1}(\psi_1(x) + \psi_2(y)) = x + y \quad (x, y \in I)$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2 : I \rightarrow \mathbb{R}$ are continuous, strictly monotone functions such that $\varphi_1, \varphi_2, \psi_1, \psi_2$ are monotone in the same sense.

In order to formulate our main result, we need the following definition:

**Definition.** Let $I \subset \mathbb{R}$ be a nonempty open interval. Let $\mathcal{D}^0(I)$ denote the class of all pairs $(\varphi_1, \varphi_2)$ of continuous functions defined on $I$ such that either $\varphi_1$ and $\varphi_2$ are strictly increasing or $\varphi_1$ and $\varphi_2$ are strictly decreasing. For $k \geq 1$, let $\mathcal{D}^k(I)$ denote the class of all those pairs $(\varphi_1, \varphi_2)$ of $k$-times continuously differentiable functions defined on $I$ such that $\varphi'_1(x)\varphi'_2(x) > 0$ for $x \in I$.

It easily follows from this definition that if $(\varphi_1, \varphi_2) \in \mathcal{D}^k(I)$ then $(\varphi_1, \varphi_2) \in \mathcal{D}^0(I)$. Under the assumption $(\varphi_1, \varphi_2) \in \mathcal{D}^0(I)$, the left hand side of the invariance equation is well-defined.
Now our main result can be formulated as follows:

**Theorem.** Let \((\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \mathcal{D}^4(I)\). Then the invariance equation holds for every \(x, y \in I\) if and only if

(i) either there exist real constants \(p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2\) with \(p \neq 0\), \(a_1 a_2 > 0\), \(c_1 c_2 > 0\) and \(a_1 c_1 = a_2 c_2\) such that, for \(x \in I\),

\[
\varphi_1(x) = a_1 e^{px} + b_1, \quad \varphi_2(x) = a_2 e^{px} + b_2,
\]

and

\[
\psi_1(x) = c_1 e^{-px} + d_1, \quad \psi_2(x) = c_2 e^{-px} + d_2;
\]

(ii) or there exist real constants \(a, b, c, d_1, d_2\) with \(ac \neq 0\) such that, for \(x \in I\),

\[
\varphi_1(x) + \varphi_2(x) = ax + b,
\]

and

\[
\psi_1(x) = c \varphi_2(x) + d_1, \quad \psi_2(x) = c \varphi_1(x) + d_2.
\]
Sufficiency

For the sufficiency part of the main theorem, we need not require the 4 times continuous differentiability of the unknown functions, therefore we have the following stronger statement.

**Theorem.** Let \((\varphi_1, \varphi_2), (\psi_1, \psi_2) \in D^0(I)\) and assume that one of the alternatives (i)–(ii) of the main theorem holds. Then the invariance equation is satisfied for every \(x, y \in I\).
4. Partial derivatives of generalized quasi-arithmetic means

To prove our main result, we will need explicite formulae for the partial derivatives of the mean $M_{\varphi}$ along the diagonal of the Cartesian product $I \times I$.

**Theorem.** If $\varphi = (\varphi_1, \varphi_2) \in D^k(I)$ then $M_{\varphi}$ is $k$-times continuously differentiable on $I \times I$ and, for all $x \in I$,

(i) if $\varphi \in D^1(I)$, then

$$\partial_1 M_{\varphi}(x, x) = \frac{\varphi_1'}{\varphi'}(x), \quad \partial_2 M_{\varphi}(x, x) = \frac{\varphi_2'}{\varphi'}(x);$$

(ii) if $\varphi \in D^2(I)$, then

$$\partial_1^2 M_{\varphi}(x, x) = \frac{\varphi_2'' \varphi_1' - \varphi_1'' \varphi_1'^2}{\varphi'^3}(x),$$

$$\partial_1 \partial_2 M_{\varphi}(x, x) = -\frac{\varphi'' \varphi_1' \varphi_2'}{\varphi'^3}(x),$$

$$\partial_2^2 M_{\varphi}(x, x) = \frac{\varphi_2'^2 \varphi_2'' - \varphi_2'' \varphi_2'^2}{\varphi'^3}(x);$$
(iii) if $\varphi \in D^3(I)$, then

$$
\partial_1^3 M_\varphi(x, x) = \frac{3\varphi''^2 \varphi_1^3 - \varphi''' \varphi' \varphi_1^3 - 3\varphi'' \varphi^2 \varphi'' \varphi_1 + \varphi^4 \varphi'''}{\varphi'^5}(x),
$$

(6) $$
\partial_1^2 \partial_2 M_\varphi(x, x) = -\frac{\varphi_2 (\varphi''' \varphi' \varphi_1^2 - 3 \varphi'' \varphi_1^2 + \varphi'' \varphi_1^2 \varphi_2'')}{\varphi'^5}(x),
$$

$$
\partial_1 \partial_2^2 M_\varphi(x, x) = -\frac{\varphi_1 (\varphi''' \varphi' \varphi_2^2 - 3 \varphi'' \varphi_2^2 + \varphi'' \varphi_2^2 \varphi_1'')}{\varphi'^5}(x),
$$

$$
\partial_2^3 M_\varphi(x, x) = \frac{3\varphi''^2 \varphi_2^3 - \varphi''' \varphi' \varphi_2^3 - 3\varphi'' \varphi^2 \varphi'' \varphi_2 + \varphi^4 \varphi'''}{\varphi'^5}(x);
$$
(iv) if $\varphi \in D^4(I)$, then

$$
\partial_1^4 M_\varphi(x, x) = \left( -\frac{1}{\varphi^7} \left( \varphi''''\varphi'^2 \varphi_1^4 - 10 \varphi''\varphi' \varphi_1^4 \\
+ 6 \varphi''' \varphi_1 \varphi_2^2 + 15 \varphi'\varphi_1^4 - 18 \varphi'' \varphi_2 \varphi_1 \varphi_2^2 \\
+ 3 \varphi'' \varphi_1^2 + 4 \varphi' \varphi_1^4 \varphi_1' - \varphi'''' \varphi_1^6 \right) \right)(x),
$$

$$
\partial_1^3 \partial_2 M_\varphi(x, x) = \left( -\frac{\varphi_1'}{\varphi^7} \left( \varphi'''' \varphi_1^3 - 10 \varphi'' \varphi_1 \varphi_2^3 + 3 \varphi''' \varphi_1 \varphi_2^3 \\
+ 15 \varphi''' \varphi_1^3 - 9 \varphi'' \varphi_1^2 \varphi_2 + \varphi'' \varphi_1^4 \varphi_2^3 \right) \right)(x),
$$

$$
\partial_1^2 \partial_2^2 M_\varphi(x, x) = \left( -\frac{1}{\varphi^7} \left( \varphi'''' \varphi_1^2 \varphi_2^2 - 10 \varphi'' \varphi_1 \varphi_2^2 \varphi_2^2 \\
+ \varphi''' \varphi_1 \varphi_2^2 + 15 \varphi' \varphi_1 \varphi_2^2 - 3 \varphi'' \varphi_1 \varphi_2^2 \varphi_2^2 \\
+ \varphi''' \varphi_1^2 \varphi_2'' - 3 \varphi'' \varphi_1 \varphi_2 \varphi_2'' + \varphi'' \varphi_1^4 \varphi_2'' \right) \right)(x),
$$

$$
\partial_1 \partial_2^3 M_\varphi(x, x) = \left( -\frac{\varphi_1'}{\varphi^7} \left( \varphi'''' \varphi_2^3 - 10 \varphi'' \varphi_2 \varphi_2^3 + 3 \varphi''' \varphi_2 \varphi_2^3 \\
+ 15 \varphi''' \varphi_2^3 - 9 \varphi'' \varphi_2 \varphi_2^3 \varphi_2 + \varphi'' \varphi_2^4 \varphi_2'' \right) \right)(x),
$$

(7)
\[ \partial_2^4 \mathcal{M}_\varphi(x, x) = \left( -\frac{1}{\varphi'^3} \left( \varphi'''^2 \varphi_2^4 \varphi'^4_2 - 10 \varphi''' \varphi'' \varphi' \varphi'^4_2 \right. \\
+ 6 \varphi''' \varphi' \varphi'^2_2 \varphi' \varphi'^2_2 + 15 \varphi'''^3 \varphi'^4_2 - 18 \varphi'' \varphi'^2 \varphi''^2_2 \varphi'^2_2 \\
+ 3 \varphi'' \varphi'^4 \varphi'^2_2 + 4 \varphi'' \varphi'^4 \varphi_2^2 \varphi'_2 - \varphi_2^4 \varphi'^6 \left) \right)(x). \]
Proof. (i) Let $\varphi \in D^1(I)$. Then $\varphi = \varphi_1 + \varphi_2$ is also continuous, strictly monotone, and hence invertible. Therefore,

$$M_\varphi(x, y) = (\varphi_1 + \varphi_2)^{-1}(\varphi_1(x) + \varphi_2(y)),$$

is well-defined for every $x, y \in I$. Since $\varphi'$ is continuous and does not vanish anywhere, it follows that $M_\varphi$ is continuously differentiable on $I \times I$. If $\varphi \in D^k(I)$ holds for some $k \geq 2$, then the $k$-times continuously differentiability of $M_\varphi$ also follows.

Using (8), we have

$$\varphi(M_\varphi(x, y)) = \varphi_1(x) + \varphi_2(y) \quad (x, y \in I).$$

By differentiating this identity with respect to the first variable,

$$\varphi'(M_\varphi(x, y)) \cdot \partial_1 M_\varphi(x, y) = \varphi'_1(x) \quad (x, y \in I).$$

Thus, taking $y := x$, the first equality in (4) follows. The second equality in (4) can be obtained by differentiating (9) with respect to the second variable.
(ii) To prove the formulae for the second-order partial derivatives in (5), let $\phi \in D^2(I)$. Then, by differentiating (10) with respect to the first variable, we get

$$\varphi''(M\varphi(x, y)) \cdot (\partial_1 M\varphi(x, y))^2 + \varphi'(M\varphi(x, y)) \cdot \partial_1^2 M\varphi(x, y) = \varphi''_1(x).$$

Therefore, taking $y := x$ and using (4),

$$\varphi''(x) \cdot \left(\frac{\varphi'_1(x)}{\varphi'(x)}\right)^2 + \varphi'(x) \cdot \partial_1^2 M\varphi(x, x) = \varphi''_1(x),$$

whence we get

$$\partial_1^2 M\varphi(x, x) = \frac{\varphi'^2 \varphi'' - \varphi''^2 \varphi'_1^2}{\varphi'^3}(x).$$

By differentiating (10) with respect to the variable $y$ and then substituting $y := x$, the second equality in (5) follows, while differentiating (9) twice with respect to the second variable, we obtain the last identity of (5).

In cases (iii) and (iv), the same argument provides the formulae for the third- and fourth-order partial derivatives.

$\square$
Necessity

First-order necessary condition

Lemma. Let \((\varphi_1, \varphi_2)\) and \((\psi_1, \psi_2)\) be pairs of class \(D^1(I)\) that satisfy the invariance equation. Then

\[
\frac{\varphi'_1}{\varphi'} = \frac{\psi'_2}{\psi'} \quad \text{and} \quad \frac{\varphi'_2}{\varphi'} = \frac{\psi'_1}{\psi'}.
\]

Proof. Differentiating the invariance equation with respect to the first variable, we get

\[
\partial_1 M_\varphi(x, x) + \partial_1 M_\psi(x, x) = 1,
\]

whence, using (4),

\[
\frac{\varphi'_1}{\varphi'} + \frac{\psi'_1}{\psi'} = 1.
\]

Therefore

\[
\frac{\varphi'_1}{\varphi'} = 1 - \frac{\psi'_1}{\psi'} = \frac{\psi' - \psi'_1}{\psi'} = \frac{\psi'_2}{\psi'}.
\]

By differentiating the invariance equation with respect to the second variable, the same calculation yields that

\[
\frac{\varphi'_2}{\varphi'} = \frac{\psi'_1}{\psi'},
\]

which makes our proof complete. \(\square\)
Second-order necessary conditions

Lemma. Let \((\varphi_1, \varphi_2)\) and \((\psi_1, \psi_2)\) be pairs of class \(D^2(I)\) that satisfy the invariance equation. Then there exists a (nonzero) real constant \(c\) such that

\begin{equation}
\varphi' \psi' = c.
\end{equation}

Proof. Differentiating the invariance equation once with respect to both variables, we get

\[
\partial_1 \partial_2 M_{\varphi}(x, x) + \partial_1 \partial_2 M_{\psi}(x, x) = 0,
\]

which, by the first-order condition and the second-order partial derivatives yields that

\[
(\varphi' \psi')' = 0.
\]
As a consequence of this and the previous lemma, we can formulate the following theorem, which describes the connection between the functions $\varphi_1$, $\varphi_2$ and $\psi_1$, $\psi_2$, respectively.

**Theorem.** Let $(\varphi_1, \varphi_2)$ and $(\psi_1, \psi_2)$ be pairs of class $\mathcal{D}^2(I)$ that satisfy the invariance equation. Then there exists a (nonzero) real constant $c$ such that

$$
\psi'_1 = \frac{c\varphi'_2}{\varphi'^2} \quad \text{and} \quad \psi'_2 = \frac{c\varphi'_1}{\varphi'^2}.
$$
Third-order necessary conditions

**Lemma.** Let \((\varphi_1, \varphi_2)\) and \((\psi_1, \psi_2)\) be pairs of class \(D^3(I)\) that satisfy the invariance equation. Then

\[
(13) \quad \left( \varphi''' \varphi' - \varphi''^2 \right) (\varphi'_1 - \varphi'_2) \varphi'_1 \varphi'_2 + \varphi'' \varphi''' (\varphi''_1 \varphi'_2 - \varphi'_1 \varphi''_2) = 0.
\]

**Proof.** Differentiating the invariance equation twice with respect to the first variable and once with respect to the second variable, we get

\[
\partial^2_1 \partial_2 M_\varphi(x, x) + \partial^2_1 \partial_2 M_\psi(x, x) = 0.
\]

Hence, applying (6) for the means \(M_\varphi\) and \(M_\psi\), we get

\[
\frac{\varphi''' \varphi'^2 \varphi''_1 \varphi''_2}{\varphi'} - 3 \frac{\varphi'' \varphi'_1^2 \varphi'_2}{\varphi'} + \frac{\varphi'' \varphi'_1 \varphi'_2}{\varphi'} + \frac{\psi''' \psi'_1^2 \psi'_2}{\psi'} - 3 \frac{\psi'' \psi'_1 \psi'_2}{\psi'} - \frac{\psi'' \psi'_1 \psi'_2}{\psi'} = 0.
\]

Using the previous calculations, we get the statement of the lemma. \(\square\)
Fourth-order necessary conditions

**Lemma.** Let \((\varphi_1, \varphi_2)\) and \((\psi_1, \psi_2)\) be pairs of class \(\mathcal{D}^4(I)\) that satisfy the invariance equation. Then

\[
\varphi''(\varphi'_1 \varphi''_2 - \varphi'_2 \varphi''_1) = 0
\]

and

\[
\varphi''' \varphi' - \varphi''^2 = 0.
\]

**Proof of the theorem**

By equality (15), it follows that

\[
\left( \frac{\varphi''}{\varphi'} \right)' = 0,
\]

which shows that there exists a constant \(p\) such that

\[
\frac{\varphi''_1 + \varphi''_2}{\varphi'_1 + \varphi'_2} = \frac{\varphi''}{\varphi'} = p.
\]

According to the value of the constant \(p\), we distinguish two cases.
Case 1: $p \neq 0$. Then $\varphi''$ does not vanish. By (14), we have that $\varphi_2'' = \frac{\varphi_1'' \varphi_2'}{\varphi_1'}$. Therefore

$$p = \frac{\varphi_1'' + \varphi_2''}{\varphi_1' + \varphi_2'} = \frac{\varphi_1'' + \frac{\varphi_1'' \varphi_2'}{\varphi_1'}}{\varphi_1' + \varphi_2'} = \frac{\varphi_1''}{\varphi_1'}.$$ 

Similarly, we can also obtain that $\frac{\varphi_2''}{\varphi_2'} = p$. Thus, $\varphi_1$ and $\varphi_2$ are solutions of the second-order linear differential equation $f'' - pf' = 0$. The general solution of this differential equation is of the form $f(x) = ae^{px} + b$, hence the first part of the theorem follows for some constants $a_1, a_2, b_1, b_2$ with $a_1a_2 > 0$.

From the second-order conditions we also have

$$\frac{\psi''}{\psi'} = -p.$$

An analogous argument shows that the other condition holds.
Case 2: $p = 0$. This means that $\varphi'' \equiv 0$ and, by the second-order conditions, $\psi'' \equiv 0$ also holds. Therefore, there exist constants $a, b, \bar{c}, d$ with $a\bar{c} \neq 0$ such that

$$\varphi(x) = ax + b \quad \text{and} \quad \psi(x) = \bar{c}x + d.$$ 

From the first-order conditions we get

$$\psi'_1 = \frac{\varphi'_2}{\varphi'} \psi' = \frac{\bar{c}}{a} \varphi'_2 \quad \text{and} \quad \psi'_2 = \frac{\varphi'_1}{\varphi'} \psi' = \frac{\bar{c}}{a} \varphi'_1$$

which means that there exist constants $d_1, d_2$ such that

$$\psi_1 = \frac{\bar{c}}{a} \varphi_2 + d_1 \quad \text{and} \quad \psi_2 = \frac{\bar{c}}{a} \varphi_1 + d_2.$$ 

Thus, with the notation $c := \bar{c}/a$, the second part of the theorem holds.


