INSTITUTES for MATHEMATICS
(Graz University of Technology)

&

INSTITUTES for MATHEMATICS
and
SCIENTIFIC COMPUTING
(University of Graz)

M. Hinze and S. Volkwein

Error estimates for abstract linear-quadratic optimal control problems using proper orthogonal decomposition

Report No. 2/2005 October, 2005

Institute for Mathematics,
Graz University of Technology
Steyrergasse 30
A-8010 Graz, Austria

Institute for Mathematics and
Scientific Computing,
University of Graz
Heinrichstrasse 36
A-8010 Graz, Austria

ERROR ESTIMATES FOR ABSTRACT LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS USING PROPER ORTHOGONAL DECOMPOSITION

M. HINZE AND S. VOLKWEIN

Abstract. In this paper we investigate POD discretizations of abstract linear-quadratic optimal control problems with control constraints. We apply the discrete technique developed in [12] and prove error estimates for the corresponding discrete controls, where we combine error estimates for the state and the adjoint system from [17, 18]. Finally, we present numerical examples that illustrate the theoretical results.

1. Introduction

Optimal control problems for nonlinear partial differential equations are often hard to tackle numerically so that the need for developing novel techniques emerges. Only very recently the reduced order approach applied to optimal control problems for partial differential equations has received an increasing amount of attention [1, 2, 3, 5, 13, 21, 23, 27, 29]. It is based on projecting the dynamical system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to, e.g., finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate.

In the present work we use proper orthogonal decomposition (POD) for deriving low order models of dynamical systems. These low order models then serve as surrogates for the dynamical system in the optimization process. We consider a linear-quadratic optimal control problem in an abstract setting and prove error estimates for the POD Galerkin approximations of the optimal control in Theorem 4.7. This is achieved by combining techniques from [7, 8, 12] and [17, 18]. The main result of the present work can be formulated as follows (see Theorem 4.7 and eq. (4.15)).

Main result: Let $\bar{u}$ denote the solution of the linear-quadratic optimal control problem, and $\bar{u}^\ell$ its POD approximation using ℓ POD basis functions for the Galerkin ansatz. Then

$$\bar{u}^\ell - \bar{u} \sim \bar{p}^\ell - \bar{p},$$

where $\bar{p} = \bar{p}(\bar{u})$ and $\bar{p}^\ell = p^\ell(\bar{u})$ denote the corresponding solutions of the continuous and discrete adjoint systems, respectively, associated to the same control $\bar{u}$. To the authors knowledge, the presented POD error estimates are the first ones for discrete controls computed by a POD Galerkin scheme.

Date: October 5, 2005.

1991 *Mathematics Subject Classification.* 35K20, 65Nxx, 90C20.

Key words and phrases. optimal control, evolution problems, proper orthogonal decomposition, error estimates.

Let us finally briefly comment on further literature containing applications of POD. The method was successfully used in a variety of fields including signal analysis and pattern recognition (see [9]), fluid dynamics and coherent structures (see [4, 14, 22, 25, 28]) and more recently in [2] to compute reduced-order controllers. The relationship between POD and balancing was considered in [19, 26, 30]. Error analysis for nonlinear dynamical systems in finite dimensions were carried out in [24].

The paper is organized as follows. In Section 2 we introduce the abstract linear-quadratic optimal control problem which is under consideration in this work. The POD Galerkin discretization is described in Section 3. Section 4 is devoted to derive the POD error estimate for the difference between the optimal solution to the abstract linear-quadratic problem and the corresponding suboptimal POD control solution. We present some numerical examples in Section 5 and, finally, we draw some conclusions in the last section.

2. The abstract linear-quadratic optimal control problems

In this section we introduce our abstract linear-quadratic optimal control problem and present first-order necessary optimality conditions. Let V and H be real separable Hilbert spaces and suppose that V is dense in H with compact embedding. By $\langle\cdot\,,\cdot\rangle_H$ we denote the inner product in H. The inner product in V is given by a symmetric bounded, coercive, bilinear form $a : V \times V \to \mathbb{R}$:

$$\langle \varphi, \psi \rangle_V = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V \tag{2.1}$$

with associated norm given by $\|\cdot\|_V = \sqrt{a(\cdot\,,\cdot)}$. Since V is continuously injected into H, there exists a constant $c_V > 0$ such that

$$\|\varphi\|_H \le c_V \; \|\varphi\|_V \quad \text{for all } \varphi \in V. \tag{2.2}$$

We associate with a the linear operator A:

$$\langle A\varphi, \psi \rangle_{V',V} = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V, \tag{2.3}$$

where $\langle\cdot\,,\cdot\rangle_{V',V}$ denotes the duality pairing between V and its dual. Then, by the Lax-Milgram lemma, A is an isomorphism from V onto V'. Due to (2.1) the operator A is the Riesz isomorphism between V and V'. Alternatively, A can be considered as a linear unbounded self-adjoint operator in H with domain

$$D(A) = \{\varphi \in V : A\varphi \in H\}.$$

By identifying H and its dual H' it follows that

$$D(A) \hookrightarrow V \hookrightarrow H = H' \hookrightarrow V',$$

each embedding being continuous and dense, when $D(A)$ is endowed with the graph norm of A.

For $T > 0$ the space $W(0,T)$ is defined as

$$W(0,T) = \big\{\varphi \in L^2(0,T;V) : \varphi_t \in L^2(0,T;V')\big\},$$

which is a Hilbert space endowed with the common inner product (see, for example, in [6, p. 473]). It is well-known that $W(0,T)$ is continuously embedded into $C([0,T];H)$, the space of continuous functions from $[0,T]$ to H, i.e., there exists an embedding constant $c_e > 0$ such that

$$\|\varphi\|_{C([0,T];H)} \le c_e \; \|\varphi\|_{W(0,T)} \quad \text{for all } \varphi \in W(0,T). \tag{2.4}$$

Let $\mathcal{U}$ be a Hilbert space which we identify with its dual $\mathcal{U}'$, and let $\mathcal{U}_{\mathsf{ad}} \subset \mathcal{U}$ a closed and convex nonempty subset. For $y_0 \in H$ and $u \in \mathcal{U}_{\mathsf{ad}}$ we consider the linear evolution problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle y(t), \varphi\rangle_H + a(y(t), \varphi) = \langle(\mathcal{B}u)(t), \varphi\rangle_{V',V} \quad \text{for } t \in [0,T],\ \varphi \in V, \tag{2.5a}$$

$$\langle y(0), \varphi\rangle_H = \langle y_0, \varphi\rangle_H \quad \text{for } \varphi \in V, \tag{2.5b}$$

where $\mathcal{B} : \mathcal{U} \to L^2(0,T;V')$ is a continuous linear operator.

Proposition 2.1. *For every $u \in \mathcal{U}$ and $y_0 \in H$ there exists a unique weak solution $y \in W(0,T)$ satisfying* (2.5) *and*

$$\|y\|_{W(0,T)} \leq C\big(\|u\|_{\mathcal{U}} + \|y_0\|_H\big) \tag{2.6}$$

with a constant $C > 0$ independent of y. If, in addition, $\mathcal{B}u \in L^2(0,T;H)$ and $y_0 \in V$ hold, then we have $y \in C([0,T];V) \cap L^2(0,T;D(A)) \cap H^1(0,T;H)$.

Proof. The proof follows by standard arguments. We refer the reader to [6], for instance. ☐

Next we introduce the cost functional $J : W(0,T) \times \mathcal{U} \to \mathbb{R}$ by

$$J(y,u) = \frac{\alpha_1}{2}\int_0^T \|y(t) - z_1(t)\|_H^2\,\mathrm{d}t + \frac{\alpha_2}{2}\|y(T) - z_2\|_H^2 + \frac{\sigma}{2}\,\|u\|_{\mathcal{U}}^2, \tag{2.7}$$

where $z_1 \in L^2(0,T;H)$ and $z_2 \in H$ are given desired states and $\alpha_1, \alpha_2, \sigma$ nonnegative parameters satisfying $\alpha_1 + \alpha_2 > 0$ and $\sigma > 0$.

Remark 2.2. Let us mention that we can replace the cost functional J given in (2.7) by the more general one

$$\tilde{J}(y,u) = \frac{\alpha_1}{2}\,\|\mathcal{C}y - z_1\|_{W_1}^2 + \frac{\alpha_2}{2}\,\|\mathcal{D}y(T) - z_2\|_{W_2}^2 + \frac{\sigma}{2}\|u\|_{\mathcal{U}}^2,$$

where W_1, W_2 are Hilbert spaces, $\mathcal{C} : L^2(0,T;H) \to W_1$ and $\mathcal{D} : H \to W_2$ are bounded linear operators, and $(z_1, z_2) \in W_1 \times W_2$ holds. In the context of (2.7) we have $W_1 = L^2(0,T;H)$, $W_2 = H$ and $\mathcal{C}$ as well as $\mathcal{D}$ are just the identity operators. To simplify the presentation we consider the cost J instead of $\tilde{J}$. ◊

The optimal control problem is given by

$$\min J(y,u) \quad \text{s.t.} \quad (y,u) \in W(0,T) \times \mathcal{U}_{\mathsf{ad}} \text{ solves (2.5)}. \tag{$\mathbf{P}$}$$

Applying standard arguments (see [20], for instance) one can prove that there exists a unique optimal solution $\bar{x} = (\bar{y}, \bar{u})$ to $(\mathbf{P})$. Moreover, there exists a unique Lagrange-multiplier $\bar{p} \in W(0,T)$ satisfying together with $\bar{x} = (\bar{y}, \bar{u})$ the *first-order necessary optimality conditions*, which consist in the *state equations* (2.5), in the *adjoint equations* in $[0,T]$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle \bar{p}(t), \varphi\rangle_H + a(\bar{p}(t), \varphi) = \alpha_1\,\langle z_1(t) - \bar{y}(t), \varphi\rangle_H \quad \text{for } t \in [0,T],\ \varphi \in V, \tag{2.8a}$$

$$\langle \bar{p}(T), \varphi\rangle_H = \alpha_2\langle z_2 - \bar{y}(T), \varphi\rangle_H \quad \text{for } \varphi \in V, \tag{2.8b}$$

and in the *optimality condition*

$$\langle \sigma\bar{u} - \mathcal{B}^\star\bar{p}, u - \bar{u}\rangle_{\mathcal{U}} \geq 0 \quad \text{for all } u \in \mathcal{U}_{\mathsf{ad}}. \tag{2.9}$$

Here, the linear and bounded operator $\mathcal{B}^\star : L^2(0,T;V) \to \mathcal{U}' \sim \mathcal{U}$ stands for the dual operator of $\mathcal{B}$ satisfying

$$\langle \mathcal{B}u, \varphi\rangle_{L^2(0,T;V'),L^2(0,T;V)} = \langle \mathcal{B}^\star\varphi, u\rangle_{\mathcal{U}} \quad \text{for all } (u,\varphi) \in \mathcal{U} \times L^2(0,T;V).$$

Recall that we have identified the Hilbert space $\mathcal{U}$ with its dual.

Introducing the so-called reduced cost functional

$$\hat{J}(u) = J(y(u), u),$$

where $y(u)$ solves (2.5) for the control $u \in \mathcal{U}_{\mathsf{ad}}$, we can express $(\mathbf{P})$ as the reduced problem

$$(\hat{\mathbf{P}}) \qquad \min \hat{J}(u) \quad \text{s.t.} \quad u \in \mathcal{U}_{\mathsf{ad}}.$$

From (2.9) it follows that the gradient of $\hat{J}'$ at $\bar{u}$ is given by

$$\hat{J}'(\bar{u}) = \sigma\bar{u} - \mathcal{B}^\star\bar{p}. \tag{2.10}$$

Let us define the operator $G : \mathcal{U} \to \mathcal{U}$ by

$$G(u) = \sigma u - \mathcal{B}^\star p, \tag{2.11}$$

where $y = y(u)$ solves the state equations (2.5) with the control $u \in \mathcal{U}$ and $p = p(y(u))$ satisfies the adjoint equations (2.8) for the state y. As a consequence of (2.9) it follows that the first-order necessary optimality conditions for a solution $u \in \mathcal{U}_{\mathsf{ad}}$ of $(\hat{\mathbf{P}})$ are given by

$$\langle G(\bar{u}), u - \bar{u}\rangle_{\mathcal{U}} \geq 0 \quad \text{for all } u \in \mathcal{U}_{\mathsf{ad}}. \tag{2.12}$$

Using a POD Galerkin scheme the operator G will be replaced by an operator $G^\ell : \mathcal{U} \to \mathcal{U}$. The construction of G^ℓ will be described in the Section 3.4.

3. The POD Galerkin discretization

In this section we introduce the POD method and derive the reduced-order model. To keep the notation simple we apply only a spatial discretization with POD basis functions, but no time integration by, e.g., an implicit Euler method. Therefore, we utilize a continuous POD. Let us mention the work [11], where convergence of POD Galerkin approximations for evolution problems were analyzed using also a continuous version of POD.

Let $u \in \mathcal{U}$ be a given control for $(\hat{\mathbf{P}})$ and $y = y(u)$ the associated state satisfying $y \in C^1([0,T];V)$.

3.1. The POD method. We define the bounded linear operator $\mathcal{Y} : H^1(0,T;\mathbb{R}) \to V$ by

$$\mathcal{Y}\varphi = \int_0^T \varphi(t)y(t) + \varphi_t(t)y_t(t)\,\mathrm{d}t \quad \text{for } \varphi \in H^1(0,T;\mathbb{R}).$$

Its Hilbert space adjoint $\mathcal{Y}^* : V \to H^1(0,T;\mathbb{R})$ satisfying

$$\langle \mathcal{Y}\varphi, z\rangle_V = \langle \varphi, \mathcal{Y}^* z\rangle_{H^1(0,T;\mathbb{R})} \quad \text{for } (\varphi, z) \in H^1(0,T;\mathbb{R}) \times V$$

is given by

$$\big(\mathcal{Y}^* z\big)(t) = \langle z, y(t) + y_t(t)\rangle_V \quad \text{for } z \in V \text{ and } t \in [0,T].$$

Furthermore, we find that the bounded linear operator $\mathcal{R} = \mathcal{Y}\mathcal{Y}^* : V \to V$ has the form

$$\mathcal{R}z = \int_0^T \langle z, y(t)\rangle_V y(t) + \langle z, y_t(t)\rangle_V y_t(t)\,\mathrm{d}t \quad \text{for } z \in V.$$

Since $y \in C^1([0,T];V)$ holds, the Kolmogorov compactness criterion in $H^1(0,T;V)$ implies that $\mathcal{Y}^* : V \to H^1(0,T;V)$ is compact. Boundedness of $\mathcal{Y}$ implies that $\mathcal{R}$ is compact as well. From the Hilbert-Schmidt theorem it follows that there exists a complete orthonormal basis $\{\psi_i\}_{i\in\mathbb{N}}$ for V and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of nonnegative real numbers so that

$$\mathcal{R}\psi_i = \lambda_i\psi_i, \quad \lambda_1 \geq \lambda_2 \geq \ldots, \quad \text{and } \lambda_i \to 0 \text{ as } i \to \infty.$$

The spectrum of $\mathcal{R}$ is a pure point spectra except for possibly 0. Each non-zero eigenvalue of $\mathcal{R}$ has finite multiplicity and 0 is the only possible accumulation point of the spectrum of $\mathcal{R}$, see [16, p. 185]. Let us note that

$$\int_0^T \|y(t)\|_V^2\,\mathrm{d}t = \sum_{i=1}^\infty \lambda_i \quad \text{and} \quad \|y_0\|_V^2 = \sum_{i=1}^\infty \big|\langle y_0, \psi_i\rangle_V\big|^2.$$

Remark 3.1. Analogous to the theory of singular value decomposition for matrices, we find that the linear and bounded operator $\mathcal{K} = \mathcal{Y}^*\mathcal{Y} : H^1(0,T;\mathbb{R}) \to H^1(0,T;\mathbb{R})$ given by

$$\big(\mathcal{K}\varphi\big)(t) = \int_0^T \langle y(s), y(t)\rangle_V \varphi(s) + \langle y_t(s), y_t(t)\rangle_V \varphi_t(s)\,\mathrm{d}s \quad \text{for } \varphi \in H^1(0,T;\mathbb{R})$$

has the same eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ as the operator $\mathcal{R}$ and the eigenfunctions

$$v_i(t) = \frac{1}{\sqrt{\lambda_i}}\,\big(\mathcal{Y}^*\psi_i\big)(t) = \frac{1}{\sqrt{\lambda_i}}\,\langle \psi_i, y(t) + y_t(t)\rangle_V$$

for $i \in \{j \in \mathbb{N} : \lambda_j > 0\}$ and almost all $t \in [0,T]$. $\Diamond$

In the following theorem we formulate properties of the eigenvalues and eigenfunctions of $\mathcal{R}$. Therefore, for given $\ell \in \mathbb{N}$ we introduce the mapping

$$\mathfrak{J} : \underbrace{V \times \ldots \times V}_{\ell-\text{times}} \to \mathbb{R}$$

by

$$\mathfrak{J}(\psi, \ldots, \psi_\ell) = \int_0^T \Big\| y(t) - \sum_{i=1}^\ell \langle y(t), \psi_i\rangle_V \psi_i \Big\|_V^2 + \Big\| y_t(t) - \sum_{i=1}^\ell \langle y_t(t), \psi_i\rangle_V \psi_i \Big\|_V^2\,\mathrm{d}t.$$

Theorem 3.2. *Let $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ denote the eigenvalues and eigenfunctions, respectively, of $\mathcal{R}$. Then, for every $\ell \in \mathbb{N}$ the first ℓ eigenfunctions $\psi_1, \ldots, \psi_\ell \in V$ solve the minimization problem*

$$\min \mathfrak{J}(\tilde\psi_1, \ldots, \tilde\psi_\ell) \quad \text{s.t.} \quad \langle \tilde\psi_j, \tilde\psi_i\rangle_V = \delta_{ij} \quad \text{for } 1 \leq i, j \leq \ell. \tag{3.1}$$

Moreover,

$$\mathfrak{J}(\psi_1, \ldots, \psi_\ell) = \sum_{i=\ell+1}^\infty \lambda_i \quad \text{for any } \ell \in \mathbb{N}. \tag{3.2}$$

Proof. The proof of the theorem relies on the fact that the eigenvalue problem

$$\mathcal{R}\psi_i = \lambda_i \psi_i \quad \text{for } i = 1, \ldots, \ell \tag{3.3}$$

is the first-order necessary optimality condition for (3.1). For more details we refer the reader to [14, Section 3]. ☐

3.2. POD Galerkin scheme for the state equation. Let us fix $\ell \in \mathbb{N}$ and compute the first ℓ POD basis functions $\psi_1, \ldots, \psi_\ell \in V$ by solving either (3.3) or $\mathcal{K} v_i = \lambda v_i$ for $i = 1, \ldots, \ell$ (see Remark 3.1). Then we define the finite dimensional linear space

$$V^\ell = \operatorname{span}\left\{\psi_1, \ldots, \psi_\ell\right\} \subset V.$$

Endowed with the topology in V it follows that V^ℓ is a Hilbert space. Next we introduce the orthogonal projection $\mathcal{P}^\ell$ of V onto V^ℓ by

$$\mathcal{P}^\ell \varphi = \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \psi_i \quad \text{for } \varphi \in V. \tag{3.4}$$

Note that

$$\mathfrak{J}(\psi, \ldots, \psi_\ell) = \int_0^T \left\| y(t) - \mathcal{P}^\ell y(t) \right\|_V^2 + \left\| y_t(t) - \mathcal{P}^\ell y_t(t) \right\|_V^2 \mathrm{d}t = \sum_{i=\ell+1}^{\infty} \lambda_i, \tag{3.5}$$

where $\mathfrak{J}$ has been introduced in Section 3.1. From (2.1), (3.4) and $\mathcal{P}^\ell \psi = \psi$ for all $\psi \in V^\ell$ it follows that

$$\begin{aligned} a(\mathcal{P}^\ell \varphi, \psi) = \langle \mathcal{P}^\ell \varphi, \psi \rangle_V &= \left\langle \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \psi_i, \psi \right\rangle_V = \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \langle \psi, \psi_i \rangle_V \\ &= \left\langle \varphi, \sum_{i=1}^{\ell} \langle \psi, \psi_i \rangle_V \psi_i \right\rangle_V = \langle \varphi, \mathcal{P}^\ell \psi \rangle_V = \langle \varphi, \psi \rangle_V = a(\varphi, \psi) \end{aligned}$$

for all $\varphi \in V$ and $\varphi \in V$. Clearly, we have $\|\mathcal{P}^\ell\|_{L(V)} = 1$, where $L(V)$ denotes the Banach space of all bounded linear operators from V into itself.

The POD Galerkin scheme for the state equation (2.5) leads to the following linear problem: determine a function $y^\ell \in C([0,T]; V^\ell)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle y^\ell(t), \psi \rangle_H + a(y^\ell(t), \psi) = \langle (\mathcal{B}u)(t), \psi \rangle_{V',V} \quad \text{for } t \in [0,T],\ \psi \in V^\ell, \tag{3.6a}$$

$$\langle y^\ell(0), \psi \rangle_H = \langle y_0, \psi \rangle_H \quad \text{for } \psi \in V^\ell. \tag{3.6b}$$

Since $a(\cdot\,, \cdot)$ is a symmetric coercive bilinear form and V^ℓ has finite dimension it follows that there exists a unique solution y^ℓ to (3.6). From $\psi = \mathcal{P}^\ell \psi$ for any $\psi \in V^\ell$ and (3.6b) we infer that

$$\begin{aligned} \langle y^\ell(0), \psi \rangle_H = \langle y_0, \psi \rangle_H &= \left\langle y_0, \sum_{i=1}^{\ell} \langle \psi, \psi_i \rangle_V \psi_i \right\rangle_H = \sum_{i=1}^{\ell} \langle y_0, \psi_i \rangle_H \langle \psi, \psi_i \rangle_V \\ &= \langle \psi, \mathcal{T}^\ell y_0 \rangle_V = \langle \mathcal{T}^\ell y_0, \psi \rangle_V \quad \text{for all } \psi \in V^\ell \end{aligned}$$

where the bounded linear operator $\mathcal{T}^\ell : H \to V^\ell$ is given by

$$\mathcal{T}^\ell \phi = \sum_{i=1}^{\ell} \langle \phi, \psi_i \rangle_H \psi_i \quad \text{for } \phi \in H.$$

Hence, we have $y^\ell(0) = \mathcal{T}^\ell y_0$ in $V^\ell \subset H$ and for $y_0 \in V$ we derive

$$y^\ell(0) - \mathcal{P}^\ell y_0 = (\mathcal{T}^\ell - \mathcal{P}^\ell) y_0. \tag{3.7}$$

Remark 3.3. Let Y be the closure of the space $\operatorname{Span}\{\psi_i\}_{i\in\mathbb{N}}$ in H, i.e., $Y = \overline{\operatorname{Span}\{\psi_i\}_{i\in\mathbb{N}}}^H$. If $y_0 \in Y$ holds, then we have

$$y_0 = \sum_{i=1}^\infty \langle y_0, \psi_i\rangle_H \psi_i$$

so that

$$\lim_{\ell\to\infty} \|\mathcal{T}^\ell y_0\|_H^2 = \lim_{\ell\to\infty} \Bigg\| \sum_{i=\ell+1}^\infty \langle y_0, \psi_i\rangle_H \psi_i \Bigg\|_H^2 = 0. \tag{3.8}$$

Since V is separable, we have $y_0 = \sum_{i=1}^\infty \langle y_0, \psi_i\rangle_V \psi_i$ and, therefore,

$$\lim_{\ell\to\infty} \|\mathcal{P}^\ell y_0\|_H^2 \le c_V^2 \lim_{\ell\to\infty} \|\mathcal{P}^\ell y_0\|_V^2 = \sum_{i=\ell+1}^\infty \big|\langle y_0, \psi_i\rangle_V\big|^2 = 0 \tag{3.9}$$

for any $y_0 \in V$, in particular, for any $y_0 \in Y \subset V$. Combining (3.8) and (3.9) we obtain $\lim_{\ell\to\infty} \|(\mathcal{T}^\ell - \mathcal{P}^\ell)y_0\|_H^2 = 0$. ♢

The following proposition ensures existence of a unique solution to (3.6) that is bounded independent of ℓ.

Proposition 3.4. *For every $u \in \mathcal{U}$ and $y_0 \in H$ (3.6) for every $\ell \in \mathbb{N}$ admits a unique solution $y^\ell \in W(0,T)$ which satisfies the estimate*

$$\|y^\ell\|_{W(0,T)} \le C\,(\|u\|_{\mathcal{U}} + \|y_0\|_H) \quad \textit{for all } \ell \in \mathbb{N}$$

with a constant $C > 0$.

Proof. It follows by variational arguments that there exists a unique solution to (3.6) satisfying

$$\|y^\ell\|_{L^\infty(0,T;H)} + \|y^\ell\|_{L^2(0,T;V)} \le C\,(\|u\|_{\mathcal{U}} + \|y_0\|_H) \quad \text{for all } \ell \in \mathbb{N}$$

and for a constant $C > 0$ independent of ℓ. In (2.3) we have introduced the linear operator $A : V \to V'$. For $y^\ell \in L^2(0,T;V)$ and $u \in \mathcal{U}$ we obtain $(-Ay^\ell + \mathcal{B}u)(t) \in V'$ for almost all $t \in [0,T]$. If y_t^ℓ denotes the distributional derivative of y^ℓ, then we have

$$y_t^\ell = -Ay^\ell + \mathcal{B}u \in L^2(0,T;V').$$

Now, $y^\ell \in W(0,T)$ and the proposed estimate follow. ☐

From Proposition 3.4 it follows that y^ℓ is bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$ by a constant independent of ℓ. This is used in [11] to prove convergence properties of the sequence $\{y^\ell\}_{\ell\in\mathbb{N}}$.

3.3. POD Galerkin scheme for the adjoint equation. Now we turn to the POD Galerkin scheme for the adjoint system (2.8a). For that purpose let $u \in \mathcal{U}$ be arbitrarily given. and let $y^\ell \in W(0,T) \cap C([0,T];V^\ell)$ denote the unique solution to (3.6). Then, $p^\ell \in C([0,T];V^\ell)$ satisfies the linear system

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle p^\ell(t),\psi\rangle_H + a(p^\ell(t),\psi) = \alpha_1 \left\langle (z_1 - y^\ell)(t),\psi\right\rangle_H,\ t \in [0,T],\ \psi \in V^\ell, \tag{3.10a}$$

$$\langle p^\ell(T),\psi\rangle_H = \alpha_2 \langle z_2 - y^\ell(T),\psi\rangle_H, \quad \psi \in V^\ell. \tag{3.10b}$$

Analogous to the arguments for the solvability of (3.6) it follows that there exists a unique solution $p^\ell \in C([0,T];V^\ell)$ to (3.10). Moreover, we have the following proposition which can be proved analogous to Proposition 3.4.

Proposition 3.5. *For every $u \in \mathcal{U}$ and $y_0 \in H$ let $y^\ell \in W(0,T) \cap C([0,T];V^\ell)$ denote the unique solution to* (3.6)*. Then,* (3.10) *for every $\ell \in \mathbb{N}$ admits a unique solution $p^\ell \in W(0,T)$ which satisfies the estimate*

$$\|p^\ell\|_{W(0,T)} \le C\left(\|u\|_{\mathcal{U}} + \|y_0\|_H + \|z_1\|_{L^2(0,T;H)} + \|z_2\|_H\right) \quad \textit{for all } \ell \in \mathbb{N}$$

with a constant $C > 0$ independent of ℓ.

3.4. POD approximation of the operator G. Now we define the approximation $G^\ell : \mathcal{U} \to \mathcal{U}$ of the operator G by

$$G^\ell(u) = \sigma u - \mathcal{B}^\star p^\ell, \tag{3.11}$$

where, $y^\ell, p^\ell \in W(0,T)$ are the unique solutions to (3.6) and (3.10), respectively. By Propositions 3.4 and 3.5 it follows that the operator G^ℓ is well-defined. Then, instead of (2.12) we consider the inequality

$$\langle G^\ell(\bar u^\ell), u - \bar u^\ell\rangle_{\mathcal{U}} \ge 0 \quad \text{for all } u \in \mathcal{U}_{\mathsf{ad}},$$

which are the first-order necessary optimality conditions for the optimal control problem

$$\min \hat J^\ell(u) \quad \text{s.t.} \quad u \in \mathcal{U}_{\mathsf{ad}}, \tag{$\hat{\mathbf{P}}^\ell$}$$

where $\hat J^\ell(u) = J(y^\ell(u),u)$ and $y^\ell(u)$ for $u \in \mathcal{U}_{\mathsf{ad}}$ denotes the solution to (3.6). We say that $(\hat{\mathbf{P}}^\ell)$ is the reduced-order model for $(\hat{\mathbf{P}})$.

Remark 3.6. We recall, that the set of admissible controls is not discretized a-priori. The discretization of the optimal control u^ℓ is determined by that of the corresponding Lagrange multiplier p^ℓ. For details of this discrete concept we refer to [12]. ◊

4. POD error estimates

In this section we derive an error estimate for the term $\|\bar u - \bar u^\ell\|_{\mathcal{U}}$, where $\bar u$ and $\bar u^\ell$ denote the unique solutions to $(\hat{\mathbf{P}})$ and $(\hat{\mathbf{P}}^\ell)$, respectively. First let us define the linear operator $\mathcal{S} : L^2(0,T;V') \times H \to W(0,T)$ by $y = \mathcal{S}(f,\phi)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y(t),\varphi\rangle_H + a(y(t),\varphi) = \langle f(t),\psi\rangle_{V',V} \qquad \text{for } t \in [0,T],\ \varphi \in V, \tag{4.1a}$$

$$\langle y(0),\varphi\rangle_H = \langle \phi,\varphi\rangle_H \qquad \text{for } \varphi \in V. \tag{4.1b}$$

Due to Proposition 2.1 there exists a unique solution $y = \mathcal{S}(f,\phi) \in W(0,T)$ to (4.1) for any $(f,\phi) \in L^2(0,T;V') \times H$, i.e., the operator $\mathcal{S}$ is well-defined. Due to

(2.6) the mapping $\mathcal{S}$ is bounded. We denote by $\mathcal{S}^\star : W(0,T)' \to L^2(0,T;V) \times H$ the dual operator associated with $\mathcal{S}$. Then we have

$$\langle r, \mathcal{S}(f,\phi)\rangle_{W(0,T)',W(0,T)} = \langle (f,\phi), \mathcal{S}^\star r\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H} \tag{4.2}$$

for all $(r,f,\phi) \in W(0,T)' \times L^2(0,T;V') \times H$. We will make use of the following lemma.

Lemma 4.1. *Suppose that $z_1 \in L^2(0,T;H)$, $z_2 \in H$ and $y = \mathcal{S}(\mathcal{B}u, y_0) \in W(0,T)$ for given $(u,y_0) \in \mathcal{U} \times H$. Moreover, let $p \in W(0,T)$ denote the unique solution to the adjoint problem*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle p(t),\varphi\rangle_H + a(p(t),\varphi) = \alpha_1 \langle z_1(t) - y(t),\varphi\rangle_H \quad \text{for } t \in [0,T],\ \varphi \in V, \tag{4.3a}$$

$$\langle p(T),\varphi\rangle_H = \alpha_2 \langle z_2 - y(T),\varphi\rangle_H \quad \text{for } \varphi \in V. \tag{4.3b}$$

Furthermore, $\Theta : W(0,T) \to W(0,T)'$ and $\Xi : L^2(0,T;H) \times H \to W(0,T)'$ are given by

$$\langle \Theta(\psi),\phi\rangle_{W(0,T)',W(0,T)} = \int_0^T \alpha_1 \langle \psi(t),\phi(t)\rangle_H \,\mathrm{d}t + \alpha_2 \langle \psi(T),\phi(T)\rangle_H, \tag{4.4}$$

$$\langle \Xi(\psi_1,\psi_2),\varphi\rangle_{W(0,T)',W(0,T)} = \int_0^T \alpha_1 \langle \psi_1(t),\varphi(t)\rangle_H \,\mathrm{d}t + \alpha_2 \langle \psi_2,\varphi(T)\rangle_H \tag{4.5}$$

for $(\psi,\phi) \in W(0,T) \times W(0,T)$ and $(\psi_1,\psi_2,\varphi) \in L^2(0,T;H) \times H \times W(0,T)$. Then it follows that

$$\begin{pmatrix} p \\ p(0) \end{pmatrix} = -\mathcal{S}^\star\Big(\Theta\big(\mathcal{S}(\mathcal{B}u,y_0)\big) - \Xi(z_1,z_2)\Big).$$

Remark 4.2. For the choice of the cost functional $\tilde{J}$ introduced in Remark 2.2 we have to replace (4.3) by

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\langle p(t),\varphi\rangle_H + a(p(t),\varphi) &= \alpha_1 \langle \mathcal{C}(z_1(t) - y(t)),\mathcal{C}\varphi\rangle_{W_1} && \text{for } t \in [0,T],\ \varphi \in V,\\ \langle p(T),\varphi\rangle_H &= \alpha_2 \langle \mathcal{D}(z_2 - y(T)),\mathcal{D}\varphi\rangle_{W_2} && \text{for } \varphi \in V. \end{aligned}$$

Moreover, the linear operators Θ and Ξ has to be replaced by $\tilde{\Theta}$ and $\tilde{\Xi}$, respectively, given as

$$\begin{aligned} \langle \tilde{\Theta}(\psi),\phi\rangle_{W(0,T)',W(0,T)} &= \alpha_1 \langle \mathcal{C}\psi,\mathcal{C}\phi\rangle_{W_1} + \alpha_2 \langle \mathcal{D}\psi(T),\mathcal{D}\phi(T)\rangle_{W_2},\\ \langle \tilde{\Xi}(\psi_1,\psi_2),\varphi\rangle_{W(0,T)',W(0,T)} &= \alpha_1 \langle \mathcal{C}\psi_1,\mathcal{C}\varphi\rangle_{W_1} + \alpha_2 \langle \mathcal{D}\psi_2,\mathcal{D}\varphi(T)\rangle_{W_2} \end{aligned}$$

for $(\psi,\phi) \in W(0,T) \times W(0,T)$ and $(\psi_1,\psi_2,\varphi) \in L^2(0,T;H) \times H \times W(0,T)$. ◊

Proof of Lemma 4.1. Let $(f,\phi) \in L^2(0,T;V') \times H$ be chosen arbitrarily. The claim is proved if we prove

$$\begin{aligned} &-\big\langle (f,\phi), \mathcal{S}^\star(\Theta(\mathcal{S}(\mathcal{B}u,y_0)) - \Xi(z_1,z_2))\big\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H}\\ &\qquad = \langle (f,\phi), (p,p(0))\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H}. \end{aligned} \tag{4.6}$$

By assumption, we have $y = \mathcal{S}(\mathcal{B}u, y_0) \in W(0,T)$. Setting $w = \mathcal{S}(f,\phi) \in W(0,T)$ we infer from (4.2), (4.4), (4.5), (4.3)

$$\begin{aligned}
&-\big\langle (f,\phi), \mathcal{S}^\star(\Theta(\mathcal{S}(\mathcal{B}u,y_0)) - \Xi(z_1,z_2))\big\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H}\\
&= -\big\langle \Theta(\mathcal{S}(\mathcal{B}u,y_0)) - \Xi(z_1,z_2), \mathcal{S}(f,\chi)\big\rangle_{W(0,T)',W(0,T)}\\
&= -\langle \Theta(y) - \Xi(z_1,z_2), w\rangle_{W(0,T)',W(0,T)}\\
&= \int_0^T \alpha_1 \left\langle (z_1 - y)(t), w(t)\right\rangle_H \mathrm{d}t + \alpha_2 \left\langle z_2 - y(T), w(T)\right\rangle_H\\
&= \int_0^T -\langle p_t(t), w(t)\rangle_{V',V} + a(p(t),w(t))\,\mathrm{d}t + \langle p(T), w(T)\rangle_H\\
&= \int_0^T \langle w_t(t), p(t)\rangle_{V',V} + a(w(t),p(t))\,\mathrm{d}t + \langle w(0), p(0)\rangle_H\\
&= \int_0^T \langle f(t), p(t)\rangle_{V',V}\,\mathrm{d}t + \langle \phi, p(0)\rangle_H = \langle (f,\phi), (p,p(0))\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H},
\end{aligned}$$

so that (4.6) holds. ☐

Remark 4.3. Let us introduce the linear and bounded extension operator $\mathcal{E} : L^2(0,T;V') \to L^2(0,T;V') \times H$ as follows: for given $f \in L^2(0,T;V')$ we have

$$f \mapsto \mathcal{E}f = \big(f, 0\big) \in L^2(0,T;V') \times H.$$

It follows from

$$\begin{aligned}
\langle \mathcal{E}f, (\psi,\phi)\rangle_{L^2(0,T;V')\times H, L^2(0,T;V)\times H} &= \langle f, \psi\rangle_{L^2(0,T;V'),L^2(0,T;V)}\\
&= \big\langle f, \mathcal{E}^\star(\psi,\phi)\big\rangle_{L^2(0,T;V'),L^2(0,T;V)},
\end{aligned}$$

where the dual operator $\mathcal{E}^\star : L^2(0,T;V) \times H \to L^2(0,T;V)$ is the restriction $\psi = \mathcal{E}^\star(\psi,\phi)$ for any $(\psi,\phi) \in L^2(0,T;V) \times H$. Then, it follows that the composition $\mathcal{B}^\star\mathcal{E}^\star\mathcal{S}^\star$ belongs to $L(W(0,T)',\mathcal{U})$ and we derive from (2.11) and Lemma 4.1 that for any $u \in \mathcal{U}$

$$G(u) = \sigma\bar{u} + \mathcal{B}^\star\mathcal{E}^\star\mathcal{S}^\star\big(\Theta(\mathcal{S}(\mathcal{B}u,y_0)) - \Xi(z_1,z_2)\big) \in \mathcal{U}. \tag{4.7}$$

From (4.7) it follows that G is an affine linear operator. ◊

Associated with the operator $\mathcal{S}$ we define the linear operator $\mathcal{S}^\ell : L^2(0,T;V') \times H \to W(0,T)$ in such a way that $y^\ell = S^\ell(f,\phi)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y^\ell(t), \psi\rangle_H + a(y^\ell(t),\psi) = \langle f(t), \psi\rangle_{V',V} \qquad \text{for } t \in [0,T],\ \psi \in V^\ell, \tag{4.8a}$$

$$\langle y^\ell(0), \psi\rangle_H = \langle \phi, \psi\rangle_H \qquad \text{for } \psi \in V^\ell. \tag{4.8b}$$

We infer from Section 3.2 that the operator $\mathcal{S}^\ell$ is well-defined and bounded. Analogous to Lemma 4.1 we have the following result.

Lemma 4.4. *Suppose that* $z_1 \in L^2(0,T;H)$, $z_2 \in H$ *and* $y^\ell = \mathcal{S}^\ell(\mathcal{B}u,y_0) \in W(0,T)$ *for given* $(y_0,u) \in H \times \mathcal{U}$. *Moreover, let* $p^\ell \in W(0,T)$ *solve the adjoint problem*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle p^\ell(t), \psi\rangle_H + a(p^\ell(t),\psi) = \alpha_1 \left\langle (z_1 - y^\ell)(t), \psi\right\rangle_H,\ t \in [0,T],\ \psi \in V^\ell, \tag{4.9a}$$

$$\langle p^\ell(T), \psi\rangle_H = \alpha_2 \langle z_2 - y^\ell(T), \psi\rangle_H, \quad \psi \in V^\ell. \tag{4.9b}$$

Then it follows that

$$\begin{pmatrix} p^\ell \\ p^\ell(0) \end{pmatrix} = -(\mathcal{S}^\ell)^\star \left(\Theta\big(\mathcal{S}^\ell(\mathcal{B}u, y_0)\big) - \Xi(z_1, z_2)\right),$$

where the operators Θ *and* Ξ *have been already defined in* (4.4) *and* (4.5)*, respectively.*

Remark 4.5. Analogous to Remark 4.2 we have to change the right-hand sites in (4.9) if we deal with the cost $\tilde{J}$ introduced in Remark 2.2. ◊

Proof of Lemma 4.4. Due to the arguments in Section 3.2 the linear operator $\mathcal{S}^\ell$ is well-defined and bounded. Let $(f, \phi) \in L^2(0, T; V') \times H$ be chosen arbitrarily. Then the claim is proved if we prove that

$$\begin{aligned} &-\big\langle (f, \phi), (\mathcal{S}^\ell)^\star(\Theta(\mathcal{S}^\ell(\mathcal{B}u, y_0)) - \Xi(z_1, z_2))\big\rangle_{L^2(0,T;V') \times H, L^2(0,T;V) \times H} \\ &\qquad = \langle (f, \phi), (p^\ell, p^\ell(0))\rangle_{L^2(0,T;V') \times H, L^2(0,T;V) \times H}. \end{aligned} \tag{4.10}$$

By assumption, we have $y = \mathcal{S}^\ell(\mathcal{B}u, y_0) \in W(0, T)$. Setting $w = \mathcal{S}^\ell(f, \phi) \in W(0, T)$ we infer from (4.4) and (4.5)

$$\begin{aligned} &- \big\langle (f, \phi), (\mathcal{S}^\ell)^\star(\Theta(\mathcal{S}^\ell(\mathcal{B}u, y_0)) - \Xi(z_1, z_2))\big\rangle_{L^2(0,T;V') \times H, L^2(0,T;V) \times H} \\ &\qquad = -\big\langle \Theta(\mathcal{S}^\ell(\mathcal{B}u, y_0)) - \Xi(z_1, z_2), \mathcal{S}^\ell(f, \phi)\big\rangle_{W(0,T)', W(0,T)} \\ &\qquad = -\langle \Theta(y^\ell) - \Xi(z_1, z_2), w^\ell\rangle_{W(0,T)', W(0,T)} \\ &\qquad = \int_0^T \alpha_1 \langle (z_1 - y^\ell)(t), w^\ell(t)\rangle_H \,\mathrm{d}t + \alpha_2 \langle z_2 - y^\ell(T), w^\ell(T)\rangle_H \\ &\qquad = \int_0^T -\langle p^\ell_t(t), w^\ell(t)\rangle_{V',V} + a(p^\ell(t), w^\ell(t)) \,\mathrm{d}t + \langle p^\ell(T), w^\ell(T)\rangle_H \\ &\qquad = \int_0^T \langle w^\ell_t(t), p^\ell(t)\rangle_{V',V} + a(w^\ell(t), p^\ell(t)) \,\mathrm{d}t + \langle w^\ell(0), p^\ell(0)\rangle_H \\ &\qquad = \int_0^T \langle f(t), p^\ell(t)\rangle_{V',V} \,\mathrm{d}t + \langle \phi, p^\ell(0)\rangle_H \\ &\qquad = \langle (f, \phi), (p^\ell, p^\ell(0))\rangle_{L^2(0,T;V') \times H, L^2(0,T;V) \times H}, \end{aligned}$$

so that (4.10) holds. □

Remark 4.6. Analogous to Remark 4.3 we obtain the following representation for the operator G^ℓ:

$$G^\ell(u) = \sigma u - \mathcal{B}^\star \mathcal{E}^\star (S^\ell)^\star \big(\Theta(\mathcal{S}^\ell(\mathcal{B}u, y_0)) - \Xi(z_1, z_2)\big) \in \mathcal{U} \tag{4.11}$$

for any $u \in \mathcal{U}$. ◊

We introduce the two bounded linear operators $\mathcal{Q}, \mathcal{Q}^\ell : L^2(0, T; V') \times H \to L^2(0, T; V')$ by the following compositions

$$\mathcal{Q} = \mathcal{E}^\star \mathcal{S}^\star \Theta \mathcal{S} \quad \text{and} \quad \mathcal{Q}^\ell = \mathcal{E}^\star (\mathcal{S}^\ell)^\star \Theta \mathcal{S}^\ell.$$

Then, by (4.7) and (4.11) we find

$$G(u) = \sigma u - \mathcal{B}^\star \mathcal{Q}(\mathcal{B}u, y_0) + \mathcal{B}^\star \mathcal{E}^\star \mathcal{S}^\star \Xi(z_1, z_2), \tag{4.12a}$$

$$G^\ell(u) = \sigma u - \mathcal{B}^\star \mathcal{Q}^\ell(\mathcal{B}u, y_0) + \mathcal{B}^\star \mathcal{E}^\star (\mathcal{S}^\ell)^\star \Xi(z_1, z_2) \tag{4.12b}$$

for any $u \in \mathcal{U}$. Now we can prove an error estimate for the optimal controls.

Theorem 4.7. *The variational inequality*

$$\big\langle G^\ell(\bar u^\ell), u-\bar u^\ell\big\rangle_{\mathcal U}\geq 0\quad\textit{for all } u\in\mathcal U_{\mathsf{ad}} \tag{4.13}$$

admits a unique solution $\bar u^\ell\in\mathcal U_{\mathsf{ad}}$*, which together with the unique solution* $\bar u$ *of* (2.12) *satisfies the estimate*

$$\|\bar u-\bar u^\ell\|_{\mathcal U}\leq\frac{1}{\sigma}\Big(\|\mathcal B^\star(\mathcal Q^\ell-\mathcal Q)(\mathcal B\bar u,y_0)\|_{\mathcal U}+\|\mathcal B^\star\mathcal E^\star(\mathcal S^\star-(\mathcal S^\ell)^\star)\Xi(z_1,z_2)\|_{\mathcal U}\Big). \tag{4.14}$$

Proof. Since (4.13) represents the first-order necessary optimality condition for problem $(\hat{\mathbf P}^\ell)$, the variational inequality admits a unique solution $u^\ell\in\mathcal U_{\mathsf{ad}}$. The unique solution $\bar u$ of (2.12) therefore is admissible as test function in (4.13), as is $\bar u^\ell$ in (2.12). Hence, we have

$$\langle G^\ell(\bar u^\ell),\bar u-\bar u^\ell\rangle_{\mathcal U}\geq 0\quad\text{and}\quad\langle G(\bar u),\bar u^\ell-\bar u\rangle_{\mathcal U}\geq 0.$$

Adding both inequalities leads to

$$\langle G^\ell(\bar u^\ell)-G(\bar u),\bar u-\bar u^\ell\rangle_{\mathcal U}\geq 0.$$

Then, from (4.12) it follows that

$$\begin{aligned}\sigma\,\|\bar u-\bar u^\ell\|^2_{\mathcal U}&\leq\langle\mathcal B^\star\mathcal E^\star(\mathcal S^\ell)^\star\Theta(\mathcal S^\ell(\mathcal B\bar u^\ell,y_0))-\mathcal B^\star\mathcal E^\star\mathcal S^\star\Theta(\mathcal S(\mathcal B\bar u,y_0)),\bar u-\bar u^\ell\rangle_{\mathcal U}\\&\qquad+\langle\mathcal B^\star\mathcal E^\star\mathcal S^\star\Xi(z_1,z_2)-\mathcal B^\star\mathcal E^\star(\mathcal S^\ell)^\star\Xi(z_1,z_2),\bar u-\bar u^\ell)\rangle_{\mathcal U}\\&=\langle\mathcal B^\star(\mathcal Q^\ell-\mathcal Q)(\mathcal B\bar u,y_0),\bar u-\bar u^\ell\rangle_{\mathcal U}+\langle\mathcal B^\star\mathcal Q^\ell(\mathcal B(\bar u^\ell-\bar u),0),\bar u-\bar u^\ell\rangle_{\mathcal U}\\&\qquad+\langle\mathcal B^\star\mathcal E^\star(\mathcal S^\star-(\mathcal S^\ell)^\star)\Xi(z_1,z_2),\bar u-\bar u^\ell\rangle_{\mathcal U}.\end{aligned}$$

Due to (4.4) we have

$$\langle\Theta(\varphi),\varphi\rangle_{W(0,T)',W(0,T)}=\|\varphi\|^2_{L^2(0,T;H)}+\|\varphi(T)\|^2_H\geq 0\quad\text{for }\varphi\in W(0,T).$$

By Remark 4.3 we obtain

$$\begin{aligned}&\langle\mathcal B^\star\mathcal Q^\ell(\mathcal B(\bar u^\ell-\bar u),0),\bar u-\bar u^\ell\rangle_{\mathcal U}\\&\quad=\langle\mathcal B(\bar u-\bar u^\ell),\mathcal Q^\ell(\mathcal B(\bar u^\ell-\bar u),0)\rangle_{L^2(0,T;V'),L^2(0,T;V)}\\&\quad=\langle\mathcal E\mathcal B(\bar u-\bar u^\ell),(\mathcal S^\ell)^\star\Theta\mathcal S^\ell(\mathcal B(\bar u^\ell-\bar u),0)\rangle_{L^2(0,T;V')\times H,L^2(0,T;V)\times H}\\&\quad=\langle(\mathcal B(\bar u-\bar u^\ell),0),(\mathcal S^\ell)^\star\Theta\mathcal S^\ell(\mathcal B(\bar u^\ell-\bar u),0)\rangle_{L^2(0,T;V')\times H,L^2(0,T;V)\times H}\\&\quad=\langle\Theta\mathcal S^\ell(\mathcal B(\bar u^\ell-\bar u),0),\mathcal S^\ell(\mathcal B(\bar u-\bar u^\ell),0)\rangle_{W(0,T)',W(0,T)}\\&\quad=-\langle\Theta(\psi),\psi\rangle_{W(0,T)',W(0,T)}\leq 0\end{aligned}$$

with $\psi=\mathcal S^\ell(\mathcal B(\bar u-\bar u^\ell),0)\in W(0,T)$. Finally, an application of the Cauchy-Schwarz inequality completes the proof. □

Remark 4.8. It follows from the proof of Theorem 4.7 that

$$\begin{aligned}\|\bar u-\bar u^\ell\|_{\mathcal U}\;&\leq\frac{1}{\sigma}\Big(\big\|\mathcal B^\star\mathcal E^\star\big((\mathcal S^\ell)^\star\big[\Theta\mathcal S^\ell(\mathcal B\bar u,y_0)-\Xi(z_1,z_2)\big]\\&\qquad\qquad-\mathcal S^\star\big[\Theta\mathcal S(\mathcal B\bar u,y_0)-\Xi(z_1,z_2)\big]\big)\big\|_{\mathcal U}\Big).\end{aligned}\tag{4.15}$$

Setting

$$\bar p^\ell=(S^\ell)^\star\big[\Theta\mathcal S^\ell(\mathcal B\bar u,y_0)-\Xi(z_1,z_2)\big]\quad\text{and}\quad\bar p=S^\star\big[\Theta\mathcal S(\mathcal B\bar u,y_0)-\Xi(z_1,z_2)\big]$$

we infer from (4.15) that

$$\|\bar u-\bar u^\ell\|_{\mathcal U}\leq\frac{1}{\sigma}\|\mathcal B^\star\mathcal E^\star(\bar p^\ell-\bar p)\|_{\mathcal U}.$$

Thus, the error of $\bar{u} - \bar{u}^\ell$ can be estimated in terms of the difference $\bar{p}^\ell - \bar{p}$. ◊

Due to Remark 4.8 error estimates for $p - p^\ell$ directly lead to an error estimate for $u - u^\ell$. This is the motivation for the next proposition.

Proposition 4.9. *Let $\ell \in \mathbb{N}$ with $\lambda_\ell > 0$ be fixed, $u \in \mathcal{U}_{\mathsf{ad}}$ and $y = y(u)$ and $p = p(y(u))$ the corresponding solutions of the state equations* (2.5) *and adjoint equations* (2.8) *respectively. Suppose that the POD basis of rank ℓ is computed as described in Section* 3.1. *Then there exist constants $c_y, c_p > 0$ such that*

$$\|y^\ell - y\|^2_{L^\infty(0,T;H)} + \|y^\ell - y\|^2_{L^2(0,T;V)} \le c_y \bigg(\|(\mathcal{T}^\ell - \mathcal{P}^\ell)y_0\|^2_H + \sum_{i=\ell+1}^\infty \lambda_i^\infty \bigg) \tag{4.16}$$

and

$$\begin{aligned} \|p^\ell - p\|^2_{L^2(0,T;V)} &\le c_p \bigg(\|(\mathcal{T}^\ell - \mathcal{P}^\ell)y_0\|^2_H + \sum_{i=\ell+1}^\infty \lambda_i^\infty \bigg) \\ &\qquad + c_p \bigg(\|\mathcal{P}^\ell p - p\|^2_{L^2(0,T;V)} + \|\mathcal{P}^\ell p_t - p_t\|^2_{L^2(0,T;V)} \bigg), \end{aligned} \tag{4.17}$$

where y^ℓ and p^ℓ solve (3.6) *and* (3.10), *respectively, for the chosen u inserted in* (3.6a) *and the linear operator $\mathcal{T}$ has been introduced in Section* 3.2.

Proof. Let

$$y^\ell(t) - y(t) = y^\ell(t) - \mathcal{P}^\ell y(t) + \mathcal{P}^\ell y(t) - y(t) = \vartheta(t) + \varrho(t),$$

where $\vartheta = y^\ell - \mathcal{P}^\ell y$ and $\varrho = \mathcal{P}^\ell y - y$. From (2.1)), (3.4), (3.5) and the continuous embedding $H^1(0,T;V) \hookrightarrow L^\infty(0,T;H)$ we find

$$\|\varrho\|^2_{L^\infty(0,T;H)} + \|\varrho\|^2_{L^2(0,T;V)} \le c_E \sum_{i=\ell+1}^\infty \lambda_i^\infty \tag{4.18}$$

with an embedding constant $c_E > 0$. Using (2.5) and (3.6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \vartheta(t), \psi \rangle_H + a(\vartheta(t), \psi) = \langle y_t(t) - \mathcal{P}^\ell y_t(t), \psi \rangle_H$$

for all $\psi \in V^\ell$ and almost all $t \in (0,T)$. From (2.1), (2.2) and Young's inequality it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\vartheta(t)\|^2_H + \|\vartheta(t)\|^2_V \le c_V^2 \, \|y_t(t) - \mathcal{P}^\ell y_t(t)\|^2_V. \tag{4.19}$$

Integrating (4.19) over the interval $(0,t)$, $t \in (0,T]$, and using (3.5) we arrive at

$$\|\vartheta(t)\|^2_H + \int_0^t \|\vartheta(s)\|^2_V \,\mathrm{d}s \le \|\vartheta(0)\|^2_H + c_V^2 \sum_{i=\ell+1}^\infty \lambda_i^\infty \quad \text{for almost all } t \in [0,T]$$

Thus,

$$\operatorname*{ess\,sup}_{t \in [0,T]} \|\vartheta(t)\|^2_H + \int_0^T \|\vartheta(s)\|^2_V \,\mathrm{d}s \le \|\vartheta(0)\|^2_H + c_V^2 \sum_{i=\ell+1}^\infty \lambda_i^\infty. \tag{4.20}$$

Estimates (4.18), (4.20) and eq. (3.7) imply the existence of a constant $c_y > 0$ such that (4.16) holds. We proceed by estimating the error arising from the discretization of the adjoint equations and write

$$p^\ell(t) - p(t) = p^\ell(t) - \mathcal{P}^\ell p(t) + \mathcal{P}^\ell p(t) - p(t) = \theta(t) + \rho(t),$$

where $\theta = p^\ell - \mathcal{P}^\ell p$ and $\rho = \mathcal{P}^\ell p - p$. From (2.8b) and (3.10b) we get

$$\|\theta(T)\|_H^2 \le \alpha_2^2 \left\|y^\ell(T) - y(T)\right\|_H^2 \le \alpha_2^2 \left\|y^\ell - y\right\|_{C([0,T];H)}^2.$$

Thus, applying (2.4), (4.16) and the techniques used above for the state equations, we obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in[0,T]} \|\theta(t)\|_H^2 + \int_0^T \|\theta(s)\|_V^2 \,\mathrm{d}s \\ &\qquad \le 2c_V^2 \Big(c_V^2 c_e^2 c_y \sum_{i=\ell+1}^{\infty} \lambda_i^\infty + \|p_t - \mathcal{P}^\ell p_t\|_{L^2(0,T;V)}^2 \Big).\end{aligned}$$

Hence, there exists a constant $c_p > 0$ satisfying (4.17).

□

Remark 4.10.

a) The term $\|(\mathcal{T}^\ell - \mathcal{P}^\ell)y_0\|_H^2$ expresses how good the initial condition is approximated by the POD basis. In Remark 3.3 we have shown that $\|(\mathcal{T}^\ell - \mathcal{P}^\ell)y_0\|_H^2$ tends to zero as ℓ goes to infinity. If $y_0 \equiv 0$ holds, the error of $y^\ell - y$ depends only on the sum $\sum_{i=\ell+1}^\infty \lambda_i$.
b) Essentially, the error in the discretization of the state variable is only bounded by the sum over the not modeled eigenvalues λ_i^∞ for $i > \ell$. Since the POD basis is not computed using adjoint information, the term $\mathcal{P}^\ell p - p$ in the $H^1(0,T;V)$-norm arises in the error estimate for the adjoint variables.
c) If we have already computed a second POD basis of rank $\tilde{\ell} \in \mathbb{N}$ for the adjoint variable, then we can express the term involving the difference $\mathcal{P}^{\tilde{\ell}}p - p$ by the sum over the eigenvalues corresponding to eigenfunctions, which are not used as POD basis functions in the discretization.
d) Recall that $\{\psi_i\}_{i\in\mathbb{N}}$ is a basis of V. Thus we have $p(t) = \sum_{i=1}^\infty \langle p(t), \psi_i\rangle_V \psi_i$ and, therefore,

$$\int_0^T \|p(t) - \mathcal{P}^\ell p(t)\|_V^2 \,\mathrm{d}t \le \int_0^T \sum_{i=\ell+1}^{\infty} |a(p(t), \psi_i)|^2 \,\mathrm{d}t.$$

The sum on the right-hand side converges to zero as ℓ tends to ∞. However, usually we do not have a rate of convergence result available. In numerical applications we can evaluate $\|p - \mathcal{P}^\ell p\|_{L^2(0,T;V)}$. If the term is large then we should increase ℓ and include more eigenfunctions in our POD basis. ◊

5. Numerical experiments

This section is devoted to present numerical test examples. All coding is done in MATLAB using routines from FEMLAB 2.2 package concerning finite element implementation. The given CPU times are obtained on a standard 2 GHz desktop PC.

5.1. Estimating the decay of the eigenvalues. From Proposition 4.9 we conclude that for $y_0 = 0$ the error $y^\ell - y$ can be estimated in terms of the not modeled eigenvalues, i.e.,

$$\|y^\ell - y\|^2_{L^\infty(0,T;H)} + \|y^\ell - y\|^2_{L^2(0,T;V)} \sim \sum_{i=\ell+1}^{\infty} \lambda_i$$

Suppose that the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ decay exponentially. Therefore, we make the ansatz

$$\lambda_i = \lambda_1 e^{-\alpha(i-1)} \quad \text{for } i \geq 1, \tag{5.1}$$

where we want to determine the factor $\alpha > 0$ numerically. Let X denote either the space H or the space V. Notice that

$$\begin{aligned}\frac{\|y^\ell - y\|^2_{L^2(0,T;X)}}{\|y^{\ell+1} - y\|^2_{L^2(0,T;X)}} &\sim \frac{\sum\limits_{i=\ell+1}^{\infty} \lambda_i}{\sum\limits_{i=\ell+2}^{\infty} \lambda_i} = \frac{\sum\limits_{i=\ell+1}^{\infty} e^{-\alpha(i-1)}}{\sum\limits_{i=\ell+2}^{\infty} e^{-\alpha(i-1)}} = \frac{e^{-\alpha\ell} + e^{-\alpha(\ell+1)} + \ldots}{e^{-\alpha(\ell+1)} + e^{-\alpha(\ell+2)} + \ldots}\\ &= \frac{e^{\alpha\ell}\big(e^{-\alpha\ell} + e^{-\alpha(\ell+1)} + \ldots\big)}{e^{\alpha\ell}\big(e^{-\alpha(\ell+1)} + e^{-\alpha(\ell+2)} + \ldots\big)} = \frac{\sum\limits_{i=0}^{\infty} \big(e^{-\alpha}\big)^i}{\sum\limits_{i=0}^{\infty} \big(e^{-\alpha}\big)^i - 1} = \frac{\frac{1}{1-e^{-\alpha}}}{\frac{e^{-\alpha}}{1-e^{-\alpha}}}\\ &= e^{\alpha}\end{aligned}$$

Thus, we have

$$Q(\ell) = \ln \frac{\|y^\ell - y\|^2_{L^2(0,T;X)}}{\|y^{\ell+1} - y\|^2_{L^2(0,T;X)}} \sim \alpha, \tag{5.2}$$

and we may introduce the *experimental order of decay (EOD)* as

$$EOD := \frac{1}{\ell_{\mathsf{max}}} \sum_{k=1}^{\ell_{\mathsf{max}}} Q(k) \tag{5.3}$$

so that $EOD \approx \alpha$.

Run 1. In this example we consider the linear heat equation on the time interval $(0,T)$ in a spatial three-dimensional geometry:

$$y_t(t,\mathrm{x}) - \Delta y(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in Q = (0,T)\times\Omega, \tag{5.4a}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_1 = (0,1)\times\Gamma_1, \tag{5.4b}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = 100g(t,x) \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_2 = (0,1)\times\Gamma_2, \tag{5.4c}$$

$$y(0,\mathrm{x}) = 0 \qquad \text{for all } \mathrm{x} = (x,y,z) \in \Omega \subset \mathbb{R}^3. \tag{5.4d}$$

The bounded domain Ω is given by a cylinder between the planes $z = 0$ and $z = 0.5$ with an annulus as floor space whose rotational axis is the z-axis. The inner radius is equal to 0.4 and the outer radius radius is chosen as 1.0, see Figure 5.1. We suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and the boundary Γ_1 is the upper annulus. Furthermore, the

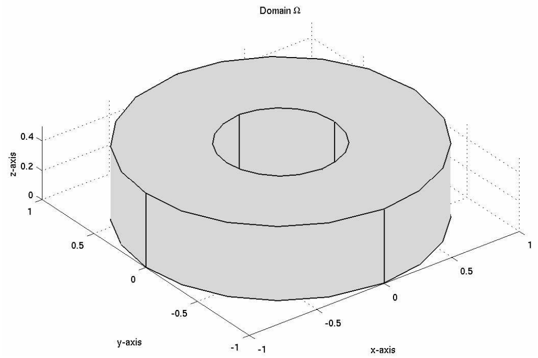

FIGURE 5.1. Run 1: Domain $\Omega \subset \mathbb{R}^3$

initial condition in (5.4d) is chosen to be $y_0 \equiv 0$ and the inhomogeneous boundary condition in (5.4c) is given by

$$g(t, \mathrm{x}) = \exp\Big(- \big[(x - 0.7\cos(2\pi t))^2 + (y - 0.7\sin(2\pi t))^2\big]\Big) \quad \text{for } (t, \mathrm{x}) \in \Sigma_2, \tag{5.5}$$

see Figure 5.2. The same geometry was used in [10] where optimal control problems for a parabolic bilinear reaction-diffusion system were considered. Note that g corresponds to a nozzle circling for $t \in [0, 1]$ once around in counter-clockwise direction at a radius of 0.7. For fixed t, the function g decays exponentially with the square of the distance from the current location of the nozzle. The 'triangulation' of the domain is created using an initial mesh obtained from the FEMLAB routine `meshinit(fem,'Hmax',0.25)` and refined by calling the FEMLAB routine `meshrefine(fem)`. The final mesh consists of 2100 degrees of freedom. In the time direction, we use $T = 1$ and partition the interval into $m = 499$ subintervals of equal lengths:

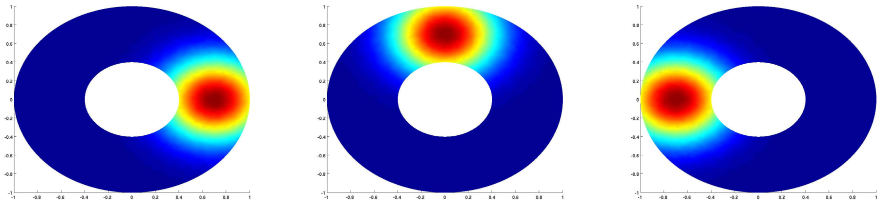

FIGURE 5.2. Run 1: Inhomogeneity $g(t, \mathrm{x})$ at $t = 0.0$, $t = 0.25$, $t = 0.5$.

$$t_j = j\Delta t, \quad j = 0, \ldots, m, \quad \Delta t = T/m. \tag{5.6}$$

First we solve (5.4) by using the implicit Euler method as time discretization, and piecewise linear, continuous finite elements (FE) for the discretization of the spatial variables. The resulting linear systems are solved by a preconditioned conjugate gradient method (MATLAB routine `pcg.m`) with an incomplete Cholesky factorization of the coefficient matrix (MATLAB routine `cholinc.m`) with the drop tolerance 10^{-6} as preconditioner. The FE solve needs 48 seconds for the run time, excluding the generation of the mesh, pre-computing integrals and the incomplete Cholesky factorization.

Here we have $H = L^2(\Omega)$ and $V = H^1(\Omega)$. To compute the POD basis we use instead of the operator $\mathcal{K}$ introduced in Remark 3.1 an approximation of $\mathcal{K}$, compare [18]. Let $\{y^h(t_j)\}_{j=0}^m$ denote the FE solution to (5.4) at the time instances t_j, $j = 0, \ldots, m$. Then we approximate the time derivatives $\dot{y}^h(t_j)$ by finite differences and set

$$y_j = \begin{cases} y^h(t_{j-1}) & \text{for } 1 \le j \le m+1, \\ \dfrac{y^h(t_{j-m-1}) - y^h(t_{j-m-2})}{\Delta t} & \text{for } m+2 \le j \le 2m+1. \end{cases}$$

Next we introduce the symmetric, positive semi-definite $(2m+1) \times (2m+1)$ matrix $\mathcal{K}^m$ with the elements $\mathcal{K}^m_{ij} = \Delta t \, \langle y_j, y_i \rangle_V$ for $1 \le i, j \le 2m+1$. In this example we have $\mathcal{K}^m \in \mathbb{R}^{999 \times 999}$. Let $\ell_{\mathsf{max}} = 15$ be chosen. For any $\ell = 1, \ldots, \ell_{\mathsf{max}}$ we compute the non-negative eigenvalues $\{\lambda_i^m\}_{i=1}^\ell$ and the corresponding eigenvectors $\{v_i^m\}_{i=1}^\ell$ of $\mathcal{K}^m$ by using the MATLAB routine `eigs.m`. The super index indicates that both the λ_i^m's and the ψ_i^m's depend on the time grid $\{t_j\}_{j=0}^m$. Notice that we have normalized all eigenvalues of $\mathcal{K}^m$ so that $\sum_{i=1}^{2m+1} \lambda_i^m = 1$ holds. The decay of the first ℓ_{max} eigenvalues is shown in Figure 5.3 (left). If $\lambda_\ell^m > 0$ holds, the POD basis $\{\psi_i^m\}_{i=1}^\ell$ of rank ℓ is given by

$$\psi_i = \frac{1}{\sqrt{\lambda_i^m}} \sum_{j=1}^{2m+1} (v_i^m)_j y_j, \quad i = 1, \ldots, \ell,$$

where $(v_i^m)_j$ denotes the j-th components of the i-th eigenvector of $\mathcal{K}^m$ for $1 \le i \le \ell$ and $1 \le j \le 2m+1$, see [18], for instance. Let

$$E(\ell) = \sum_{i=1}^{\ell} \lambda_i \cdot 100\%. \tag{5.7}$$

Then, we find $E(1) \approx 47\%$, $E(2) \approx 68\%$, $E(3) \approx 86\%$, $E(4) \approx 96\%$, and $E(5) \approx 99\%$. Using the POD basis we discretize (5.4) in the spatial variables by a Galerkin scheme and apply the implicit Euler method for the time integration. In this way we obtain POD solutions y^ℓ to (5.4) for $\ell = 1, \ldots, \ell_{\mathsf{max}}$. By Proposition 3.4 the solution y^ℓ belongs to $L^2(0,T;H^1(\Omega)) \subset L^2(0,T;L^2(\Omega))$. In Figure 5.3 (right) the corresponding norms of the difference $y^\ell - y^h$ are presented for $\ell = 1, \ldots, \ell_{\mathsf{max}}$. Of course, both norms decay with respect to the number ℓ of POD basis functions. Due to $\|\varphi\|_{L^2(\Omega)} \le \|\varphi\|_{H^1(\Omega)}$ for every $\varphi \in H^1(\Omega)$ the $L^2(0,T;L^2(\Omega))$-norm of the difference is smaller than the corresponding $L^2(0,T;H^1(\Omega))$-norm. For the experimental order of decay we obtain $EOD = 0.6683$ if we take the $L^2(0,T;L^2(\Omega))$-norm for the difference $y^\ell - y^h$, and $EOD = 0.5726$ provided we measure the difference $y^\ell - y^h$ in the $L^2(0,T;H^1(\Omega))$-norm. As Figure 5.3 (left) shows the ansatz $\lambda_i = \lambda_1 e^{-\alpha(i-1)}$, $i = 1, \ldots, \ell_{\mathsf{max}}$, reflects very well the decay of the eigenvalues.

In Table 5.1 we present the needed CPU times for the computations. As Table 5.1 shows, the POD solve needs significantly less CPU time compared to the FE solve. Of course, computing the POD basis functions and the reduced-order model takes nearly 90 seconds. Of course does the advantage of the reduced order model only pop up if system (5.4) has to be solved several times for different data, say. In this

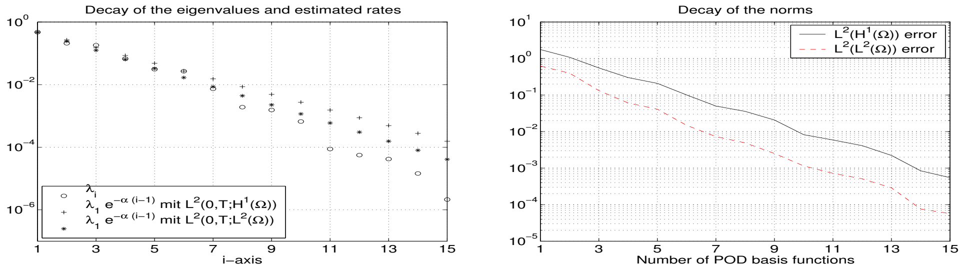

FIGURE 5.3. Run 1: Decay of the eigenvalues (left) and norms (right).

Computing the FE mesh and matrices	50.0 seconds
FE solve for (5.4)	49.9 seconds
Computing 15 POD basis functions	87.1 seconds
Computing the reduced-order model for (5.4), $\ell = 15$	$<$ 2.0 seconds
POD solve for (5.4), $\ell = 15$	$<$ 0.1 seconds

TABLE 5.1. Run 1: CPU times for the FE and POD computations.

topology	ensemble	$\\|y^h - y^\ell\\|_{L^2(0,T;H^1(\Omega))}$	$\\|y^h - y^\ell\\|_{L^2(0,T;L^2(\Omega))}$
$L^2(\Omega)$	$\{y_j\}_{i=1}^{m+1}$	0.0413	0.0045
$H^1(\Omega)$	$\{y_j\}_{i=1}^{m+1}$	0.0171	0.0015
$L^2(\Omega)$	$\{y_j\}_{i=1}^{2m+1}$	0.0168	0.0015
$H^1(\Omega)$	$\{y_j\}_{i=1}^{2m+1}$	0.0131	0.0013

TABLE 5.2. Run 1: Comparison of the norms $\|y^h - y^\ell\|_{L^2(0,T;H^1(\Omega))}$ and $\|y^h - y^\ell\|_{L^2(0,T;L^2(\Omega))}$ with $\ell = 15$ for different POD strategies (i.e., choice of the topology and ensemble).

case the POD basis functions are only computed once and the POD solve for (5.4) reduces the overall CPU time significantly.

To compute the POD basis we use the topology in the space $V = H^1(\Omega)$ and include also the difference quotients into the ensemble. In Table 5.2 the errors of the difference $y^\ell - y$ for $\ell = 15$ are compared for different POD strategies. It turns out that taking the H^1-topology and including the difference quotients lead to the smallest error in the computation. $\Diamond$

Run 2. In the second example we consider a two-dimensional problem:

$$y_t(t,\mathrm{x}) - \Delta y(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in Q = (0,T) \times \Omega, \tag{5.8a}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_1 = (0,1) \times \Gamma_1, \tag{5.8b}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = 5q(t,x) \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_2 = (0,1) \times \Gamma_2, \tag{5.8c}$$

$$y(0,\mathrm{x}) = 0(\mathrm{x}) \qquad \text{for all } \mathrm{x} = (x,y) \in \Omega \subset \mathbb{R}^2. \tag{5.8d}$$

The bounded domain Ω together with its triangulation is shown in Figure 5.4. The

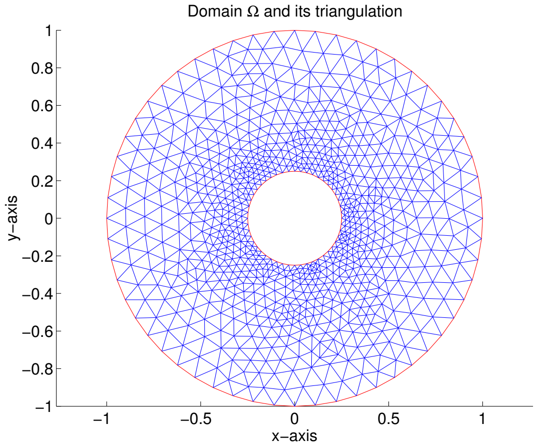

Figure 5.4. Run 2: Domain $\Omega \subset \mathbb{R}^2$ and its triangulation with triangles.

boundaries are given by

$$\Gamma_1 = \big\{\mathrm{x} \in \partial\Omega \,\big|\, \|\mathrm{x}\| = 1\big\} \quad \text{and} \quad \Gamma_2 = \partial\Omega \setminus \Gamma_1.$$

The mesh is generated as in Run 1. For this domain, the final mesh consists in 868 degrees of freedom. The partition of the time interval is the same as in the previous run, see (5.6). The boundary function q is given by

$$q(t,\mathrm{x}) = \exp(-((x-0.7\cos(2\pi t))^2 + (y-0.7\sin(2\pi t))^2)) \quad \text{for } (t,\mathrm{x}) \in \Sigma_2.$$

Again we have $H = L^2(\Omega)$ and $V = H^1(\Omega)$. Let y^h denote the finite element solution to (5.8) using piecewise linear, continuous finite elements for the approximation of the spatial variables. To compute the POD basis we proceed as in Run 1. The eigenvalues decay rapidly, compare Figure 5.5 (left). In (5.7) we have introduced the quantity $E(\ell)$. For this example there holds $E(1) \approx 25\%$, $E(3) \approx 59\%$, $E(5) \approx 89\%$, $E(7) \approx 96\%$, and $E(9) \approx 99\%$, The decay of the $L^2(0,T;L^2(\Omega))$- as well as of the $L^2(0,T;H^1(\Omega))$-norms for the difference $y - y^\ell$ are shown in Figure 5.5 (right). As expected, both norms tends to zero if ℓ increases. For this example we find $EOD = 0.4762$ if we take the $L^2(0,T;L^2(\Omega))$-norm for the difference $y^h - y^\ell$, and $EOD = 0.3828$ provided we measure the difference $y^h - y^\ell$ in the $L^2(0,T;H^1(\Omega))$-norm. As Figure 5.3 (left) shows the ansatz $\lambda_i = \lambda_1 e^{-\alpha(i-1)}$,

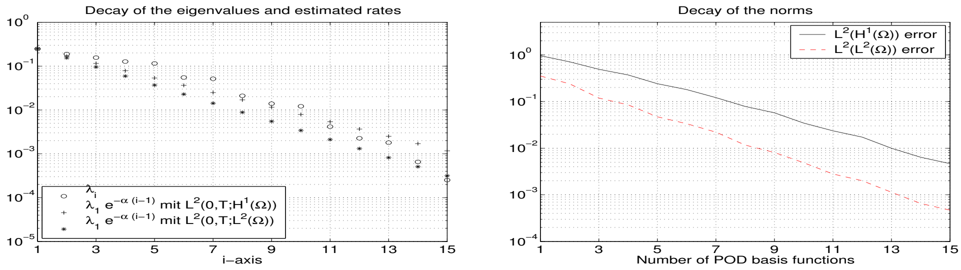

Figure 5.5. Run 2: Decay of the eigenvalues (left) and norms (right).

Computing the FE mesh and matrices	18.0 seconds
FE solve for (5.8)	5.0 seconds
Computing 15 POD basis functions	36.9 seconds
Computing the reduced-order model for (5.8), $\ell = 15$	< 0.1 seconds
POD solve for (5.4), $\ell = 15$	< 0.1 seconds

Table 5.3. Run 2: CPU times for the FE and POD computations.

topology	ensemble	$\|y^h - y^\ell\|_{L^2(0,T;H^1(\Omega))}$	$\|y^h - y^\ell\|_{L^2(0,T;L^2(\Omega))}$
$L^2(\Omega)$	$\{y_j\}_{i=1}^{m+1}$	0.0104	0.0012
$H^1(\Omega)$	$\{y_j\}_{i=1}^{m+1}$	0.0064	0.0007
$L^2(\Omega)$	$\{y_j\}_{i=1}^{2m+1}$	0.0064	0.0007
$H^1(\Omega)$	$\{y_j\}_{i=1}^{2m+1}$	0.0060	0.0006

Table 5.4. Run 2: Comparison of the norms $\|y^h-y^\ell\|_{L^2(0,T;H^1(\Omega))}$ and $\|y^h - y^\ell\|_{L^2(0,T;L^2(\Omega))}$ with $\ell = 15$ for different POD strategies (i.e., choice of the topology and ensemble).

$i = 1, \ldots, \ell_{\mathsf{max}}$, again reflects the decay of the eigenvalues, in particular for the case of the $L^2(0,T;H^1(\Omega))$-norm.

In Table 5.3 we present the needed CPU times for the computations.

To compute the POD basis we use the topology in the space $V = H^1(\Omega)$ and include also the difference quotients into the ensemble. In Table 5.4 the errors of the difference $y^\ell - y$ with $\ell = 15$ are compared for different POD strategies. As in Run 1 it turns out that taking the H^1-topology and including the difference quotients lead to the smallest error in the computation. ♢

5.2. Linear-optimal control. Now we present a numerical example that illustrates the result of Theorem 4.7. In particular, we focus on three different snapshot ensembles used to compute the POD basis. The first ensemble contains only information about the state variable, the second one only about the adjoint variables whereas the third one contains information about both variables.

Run 3. Again we set $V = H^1(\Omega)$, $H = L^2(\Omega)$, and $\mathcal{U} = L^2(0,T)$. We choose the cost functional introduced in (2.7) with $\alpha_1 = 0$, $\alpha_2 = 1$, $\sigma = 0.01$, and $z_2 = 1$. The state $y \in V$ and the control $u \in L^2(0,T)$ solve

$$y_t(t,\mathrm{x}) - \Delta y(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in Q = (0,T) \times \Omega, \tag{5.9a}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = 0 \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_1 = (0,1) \times \Gamma_1, \tag{5.9b}$$

$$\frac{\partial y}{\partial n}(t,\mathrm{x}) = u(t)g(t,x) \qquad \text{for all } (t,\mathrm{x}) \in \Sigma_2 = (0,1) \times \Gamma_2, \tag{5.9c}$$

$$y(0,\mathrm{x}) = 0 \qquad \text{for all } \mathrm{x} = (x,y,z) \in \Omega \subset \mathbb{R}^3. \tag{5.9d}$$

where the bounded domain Ω as well as the partition $\Gamma = \Gamma_1 \cap \Gamma_2$ are the same as in Run 1 and the function g is defined in (5.5).

The FE mesh is generated by the same procedure as in Run 1. So, again, we have 2100 degrees of freedom. For the time grid we take $m = 249$. All together, we obtain 1046049 unknowns for the state y, control u and adjoint variable p. The

	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$
Computing the FE mesh and matrices	38.4	38.4	38.4
FE solve for $(\hat{\mathbf{P}})$	1560.0	1560.0	1560.0
Computing 15 POD basis functions	22.4	22.5	87.0
Comp. the reduced-order model for $(\hat{\mathbf{P}})$, $\ell = 15$	0.7	0.7	0.7
POD solve for (5.4), $\ell = 15$	0.7	0.7	0.7

TABLE 5.5. Run 3: CPU times in seconds for the computation of the three POD basis function sets $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

set $\mathcal{U}_{\mathsf{ad}}$ is given by $\mathcal{U}_{\mathsf{ad}} = \{u \in L^2(0,T)\,|\,u_a \leq u \leq u_b \text{ in } (0,T) \text{ almost everywhere}\}$, where we choose $u_a = -10000$ and $u_b = 10000$. Due to the lower and upper bounds the solution $\bar{u} \in L^2(0,T)$ to $(\hat{\mathbf{P}})$ is inactive, i.e. $u_a < \bar{u} < u_b$ holds in $(0,T)$. To solve $(\hat{\mathbf{P}})$ we apply the CG method with the relative stopping criterium $\varepsilon = 10^{-8}$ for the residuum. Notice that a matrix application within each CG iteration requires a linearized state and an adjoint solve. It turns out that the discrete FE solution $\bar{u}^h$ is found after 29 CG iteration. The needed CPU time is 1560.00 seconds. We denote by $\{\bar{y}^h(t_j)\}_{j=0}^m$ and by $\{\bar{p}^h(t_j)\}_{j=0}^m$ the corresponding FE solutions to the state and adjoint equation, respectively. To compute the POD basis we compare three different snapshot ensembles:

$$\begin{aligned}
\mathcal{V}_1 &= \operatorname{span}\left\{\{\bar{y}^h(t_j)\}_{j=0}^m, \left\{\frac{\bar{y}^h(t_j) - \bar{y}^h(t_{j-1})}{\Delta t}\right\}_{j=1}^m\right\},\\
\mathcal{V}_2 &= \operatorname{span}\left\{\{\bar{p}^h(t_j)\}_{j=0}^m, \left\{\frac{\bar{p}^h(t_j) - \bar{p}^h(t_{j-1})}{\Delta t}\right\}_{j=1}^m\right\},\\
\mathcal{V}_3 &= \mathcal{V}_1 \cup \mathcal{V}_2.
\end{aligned}$$

Using the techniques described in Run 1 we compute the three POD basis function sets $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$. The needed CPU times for $\ell = 15$ are presented in Table 5.5. Since the number of members in the ensembles $\mathcal{V}_1$ and $\mathcal{V}_2$ are the same, the computational time for the POD bases nearly coincides, whereas the space $\mathcal{V}_3$ contains twice as much members as $\mathcal{V}_1$ or $\mathcal{V}_2$. In Table 5.6 we present the values for $E(\ell)$ (introduced in (5.7)) for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$. According to Remark 4.10, parts c) and d), we compare the norms $\|\bar{y}^h - \mathcal{P}^\ell \bar{y}^h\|_{L^2(0,T;X)}$ and $\|\bar{p}^h - \mathcal{P}^\ell \bar{p}^h\|_{L^2(0,T;X)}$ with $X = H^1(\Omega)$ as well as $X = L^2(\Omega)$ for the three different POD bases, compare Figures 5.6 and 5.7. Due to the choice of the snapshot ensemble, the norm $\|\bar{y} - \mathcal{P}^\ell \bar{y}\|_{L^2(0,T;X)}$ decays very slowly provided the POD basis $\{\psi_i^2\}_{i=1}^\ell$ is utilized, since only information about the adjoint variable is included in $\mathcal{V}_2$. In $\mathcal{V}_3$ the $y(t_j)^h$'s are included. Consequently, using the POD basis functions $\psi_3^1, \ldots, \psi_\ell^3$ we observe that the norm $\|\bar{y} - \mathcal{P}^\ell \bar{y}\|_{L^2(0,T;X)}$ decays faster than for the POD basis $\{\psi_i^2\}_{i=1}^\ell$. Of course, the basis functions $\psi_1^1, \ldots, \psi_\ell^1$ contain only information about the state variable $\bar{y}^h$, so that the corresponding norm $\|\bar{y} - \mathcal{P}^\ell \bar{y}\|_{L^2(0,T;X)}$ is smaller for each $\ell \in \{1, \ldots, 15\}$ compared to the same norm using the POD bases $\{\psi_i^2\}_{i=1}^\ell$ or $\{\psi_i^2\}_{i=1}^\ell$, see Figure 5.6. Analogous arguments can be

ℓ	$E(\ell)$ for $\{\psi_i^1\}_{i=1}^\ell$	$E(\ell)$ for $\{\psi_i^2\}_{i=1}^\ell$	$E(\ell)$ for $\{\psi_i^3\}_{i=1}^\ell$
$\ell = 1$	45.89 %	70.44 %	48.20 %
$\ell = 3$	87.65 %	97.41 %	84.39 %
$\ell = 5$	96.95 %	99.97 %	94.29 %
$\ell = 7$	99.37 %	100.00 %	98.06 %
$\ell = 9$	99.72 %	100.00 %	99.63 %
$\ell = 11$	99.78 %	100.00 %	99.82 %
$\ell = 13$	99.80 %	100.00 %	99.88 %
$\ell = 15$	99.80 %	100.00 %	99.90 %

TABLE 5.6. Run 3: Values for $E(\ell)$ for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

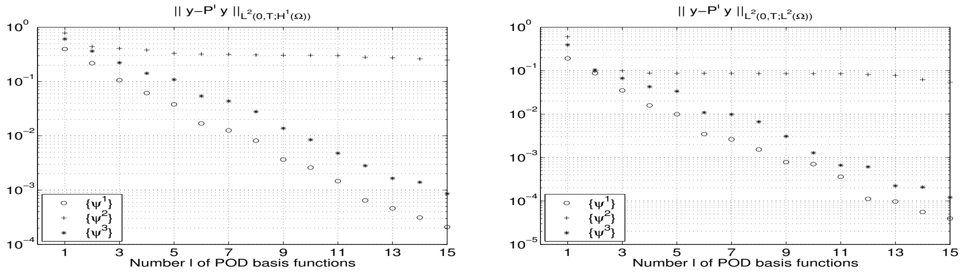

FIGURE 5.6. Run 3: Decay of the norms $\|\bar{y}^h - \mathcal{P}^\ell \bar{y}^h\|_{L^2(0,T;H^1(\Omega))}$ (left) and $\|\bar{y}^h - \mathcal{P}^\ell \bar{y}^h\|_{L^2(0,T;L^2(\Omega))}$ (right) for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

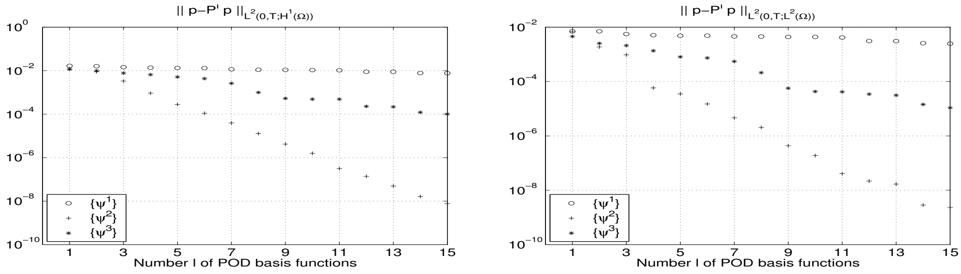

FIGURE 5.7. Run 3: Decay of the norms $\|\bar{p}^h - \mathcal{P}^\ell \bar{p}^h\|_{L^2(0,T;H^1(\Omega))}$ (left) and $\|\bar{p}^h - \mathcal{P}^\ell \bar{p}^h\|_{L^2(0,T;L^2(\Omega))}$ (right) for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

used to explain the decay of the norm $\|\bar{p}^h - \mathcal{P}^\ell \bar{p}^h\|_{L^2(0,T;X)}$ in Figure 5.7 for the three different POD bases.

In Figure 5.8 we compare the FE solution u^h with the POD Galerkin control u^ℓ for $\ell = 15$ and for the different POD bases $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$. As is shown in Figure 5.8, the Galerkin control com-

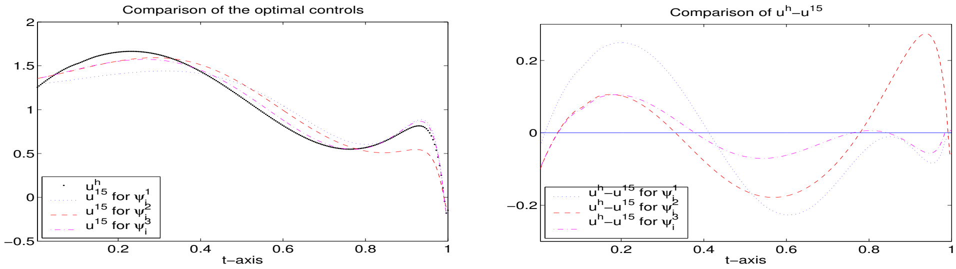

FIGURE 5.8. Run 3: Comparison of the FE optimal control u^h and the suboptimal control u^ℓ for $\ell = 15$ (left) and of the differences between u^h and u^ℓ (right) for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

ℓ	$\|u^h - u^\ell\|$ for $\{\psi_i^1\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^2\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^3\}_{i=1}^\ell$
$\ell = 1$	0.5100	0.5437	0.4672
$\ell = 3$	0.3792	0.1200	0.1869
$\ell = 5$	0.3506	0.0588	0.1201
$\ell = 7$	0.3225	0.0584	0.0676
$\ell = 9$	0.3031	0.0585	0.0566
$\ell = 11$	0.2902	0.0585	0.0557
$\ell = 13$	0.2057	0.0596	0.0555
$\ell = 15$	0.1530	0.1282	0.0555

TABLE 5.7. Run 3: Norms $\|u^h - u^\ell\|_{L^2(0,T)}$ for the different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the snapshot ensembles $\mathcal{V}_j$, $j = 1, 2, 3$.

puted with ensemble $\mathcal{V}_3$ is close to the FE solution u^h. The suboptimal control u^ℓ obtained by using the POD basis $\{\psi_i^1\}$ leads to a larger error in the time interval $(0, 0.4)$, whereas suboptimal control u^ℓ obtained by using the POD basis $\{\psi_i^2\}$ differs significantly from u^h in the time interval $(0.8, 1)$. In Table 5.7 we also present the errors between the FE optimal solution and the POD suboptimal controls in the $L^2(0,T)$-norm and for different ℓ. From Table 5.7 we conclude that including adjoint information into the snapshot ensemble is essential to obtain good approximations for the controls. In fact, the optimality conditions (2.9) directly relates the control and the adjoint variable. Let us mention that for $\ell = 15$ the norms $\|\bar{y}^h - \bar{y}^\ell\|_{L^2(0,T;H^1(\Omega))}$ are three times larger using the POD bases $\{\psi_i^j\}_{i=1}^\ell$, $j = 1, 2$ than with the POD basis built from ensemble $\mathcal{V}_3$. So it turns out, that the use of ensemble $\mathcal{V}_3$ leads to the best performance of the POD Galerkin control. This observation coincides with our estimates in Theorem 4.7 and Proposition 4.9, where the state and the adjoint variables have to be approximated well to get a small error for the difference between the optimal and the Galerkin control. ◊

Acknowledgment. The first author greatfully acknowledges the support of the collaborative research center SFB 609 sponsored by the German Research Foundation.

Further, the authors would like to thank Roland Griesse for a part of the coding.

References

[1] Afanasiev, K., Hinze, M.: Adaptive control of a wake flow using proper orthogonal decomposition. Lect. Notes Pure Appl. Math., **216**, 317–332 (2001).

[2] Atwell, J.A., Borggaard, J.T., King, B.B.: Reduced order controllers for Burgers' equation with a nonlinear observer. Int. J. Appl. Math. Comput. Sci., **11**, 1311–1330 (2001).

[3] Arian, E., Fahl, M., Sachs, E.W.: Trust-region proper orthogonal decomposition for flow control. Technical Report 2000-25, ICASE (2000).

[4] Aubry, N., Holmes, P., Lumley, J.L., Stone, E.: The dynamics of coherent structures in the wall region of a turbulent boundary layer. J. Fluid Mech., **192**, 115–173 (1988).

[5] Banks, H.T., Joyner, M.L., Winchesky, B., Winfree, W.P.: Nondestructive evaluation using a reduced-order computational methodology. Inverse Problems, **16**, 1–17 (2000).

[6] Dautray, R., Lions, J.-L.: Mathematical Analysis and Numerical Methods for Science and Technology. Volume 5: Evolution Problems I. Springer-Verlag, Berlin (1992).

[7] Deckelnick, K., Hinze, M.: Error estimates in space and time for tracking-type control of the instationary Stokes system. ISNM, **143**, 87–103 (2002).

[8] Deckelnick, K., Hinze, M.: Semidiscretization and error estimates for distributed control of the instationary Navier-Stokes equations. Numerische Mathematik, **97**, 297–320 (2004).

[9] Fukunaga, K.: Introduction to Statistical Recognition. Academic Press, New York (1990).

[10] R. Griesse and S. Volkwein. A primal-dual active set strategy for optimal boundary control of a nonlinear reaction-diffusion system. SIAM J. Control and Optimization, **44**, 467-494 (2005).

[11] Henri, T., Yvon, M.: Convergence estimates of POD Galerkin methods for parabolic problems. Technical Report No. 02-48, Institute of Mathematical Research of Rennes (2002).

[12] Hinze, M.: A variational discretization concept in control constrained optimization: the linear-quadratic case. Computational Optimization and Applications, **30**, 45-61 (2005).

[13] Hinze, M. and Kunisch, K.: Three control methods for time - dependent Fluid. Flow Flow, Turbulence and Combustion, **65**, 273-298 (2000).

[14] Holmes, P., Lumley, J.L., Berkooz, G.: Turbulence, Coherent Structures, Dynamical Systems and Symmetry. Cambridge Monographs on Mechanics, Cambridge University Press (1996).

[15] Ito, K., Ravindran, S.S.: A reduced basis method for control problems governed by PDEs. In: Desch, W., Kappel, F., Kunisch, K. (ed), Control and Estimation of Distributed Parameter Systems. Proceedings of the International Conference in Vorau, 1996, 153-168 (1998).

[16] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, Berlin (1980).

[17] Kunisch, K., Volkwein, S.: Galerkin proper orthogonal decomposition methods for parabolic problems. Numerische Mathematik, **90**, 117-148 (2001).

[18] Kunisch, K., Volkwein, S.: Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. SIAM J. Numer. Anal., **40**, 492-515 (2002).

[19] Lall, S., Marsden, J.E., Glavaski, S.: Empirical model reduction of controlled nonlinear systems. In: Proceedings of the IFAC Congress, vol. F, 473-478 (1999).

[20] Lions, J.-L.: *Optimal Control of systems governed by partial differential equations*, Springer-Verlag, Berlin, 1971.

[21] Ly, H.V., Tran, H.T.: Modeling and control of physical processes using proper orthogonal decomposition. Mathematical and Computer Modeling, **33**, 223-236, (2001).

[22] Noack, B., Afanasiev. K., Morzynski, M., Tadmor, G., Thiele, F.: A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. J. Fluid. Mech., **497**, 335-363 (2003).

[23] Ravindran, S.S.: Adaptive reduced order controllers for a thermal flow system using proper orthogonal decomposition. SIAM J. Sci. Comput., **28**, 1924-1942 (2002).

[24] Rathinam, M., Petzold, L.: Dynamic iteration using reduced order models: a method for simulation of large scale modular systems. SIAM J. Numer. Anal., **40**, 1446-1474 (2002).

[25] Rempfer, D., Fasel, H.F.: Dynamics of three-dimensional coherent structures in a flat-plate boundary layer. J. Fluid Mech. **275**, 257-283 (1994).

[26] Rowley, C.W.: Model reduction for fluids, using balanced proper orthogonal decomposition. To appear in Int. J. on Bifurcation and Chaos (2004).

[27] Shvartsman, S.Y., Kevrikidis, Y.: Nonlinear model reduction for control of distributed parameter systems: a computer-assisted study. AIChE Journal, **44**, 1579-1595 (1998).

[28] Sirovich, L.: Turbulence and the dynamics of coherent structures, parts I-III. Quart. Appl. Math., **XLV**, 561-590 (1987).

[29] Tang, K.Y., Graham, W.R., Peraire, J.: Optimal control of vortex shedding using low-order models. I: Open loop model development. II: Model based control. Int. J. Numer. Methods Eng., **44**, 945-990 (1999).

[30] Willcox, K., Peraire, J.: Balanced model reduction via the proper orthogonal decomposition. AIAA, 2001-2611 (2001).

M. Hinze, Technische Universität Dresden, Institut für Numerische Mathematik, Zellescher Weg 12-14, D-01062 Dresden, Germany
E-mail address: `hinze@math.tu-dresden.de`

S. Volkwein, Karl-Franzens-Universität Graz, Institut für Mathematik und Wissenschaftliches Rechnen, Heinrichstrasse 36, A-8010 Graz, Austria
E-mail address: `stefan.volkwein@uni-graz.at`